## EXERCISE ON CASCADES AND ON LENDING TO BET AND SHARE RISKS

(1) Consider a cascade model exactly like that we discussed in class, except that negative signals are more precise than positive signals. The notation is

$t$: indexes time, and people

$S_t$: signal observed by agent $t$, either 0 or 1

$p_H$: probability of $S_t = 1$ conditional on $H$ being true state. We assume $p_H = .7$

$p_L$: probability of $S_t = 0$ conditional on $L$ being true state. We assume $p_L = .9$

$B_t$: agent $t$ action (either "buy", $B_t = 1$, or "sell", $B_t = 0$)

Agent $t$ chooses $B_t$ based on having seen $B_s$'s for $s < t$ and $S_t$, but not $S_s$ for $s < t$. She wants to set $B_t = 1$ when $H$ is the true state, $B_t = 0$ when $L$ is the true state. Each agent, before making any observations, believes that the probabilities of the two states are equal. So agent $t$ will set $B_t = 1$ ("buy") when the conditional probability that the state is $H$, given the $S_t$ and the history $\{B_s\}$ for $s = 1, \dots, t-1$, is above .5. When the conditional probability is .5, the agent randomizes, choosing either value of $B_t$ with equal probability.

(a) What is the probability that there is a cascade starting with the very first observation — i.e. that $B_s = B_1$ for all $s$? Calculate the probably for each of the two true states, $H$ and $L$, and explain how you know that the choice $B_s$ will remain constant forever in the cases where you claim this is true.

(b) Calculate the probability of a false cascade starting with the first observation, that is $B_t = 1$ forever when the state is $L$ or $B_t = 0$ forever when the state is $H$.

(c) What is the probability that $B_3 = B_s$ for all $s > 3$? That is, what is the probability of a cascade from time 4 onward? (This should be larger than or equal to the probability of a cascade starting with $B_1$.)

First note that before a cascade starts, $B_t = S_t$. Once we have a history of $B_s, s < t$ such that $B_t$ takes on the same value regardless of $S_t$'s value, we are permanently in a cascade, as every subsequent agent has the same information. If $B_1 = 1$ and $S_2 = 1$, obviously $B_2 = 1$. But also if $B_1 = 1$ and $S_2 = 0$, $B_2 = 1$. This is because the probability of the sequence $S_1 = 1$, $S_2 = 0$ is .21 in state $H$ and .09 in state $L$, so that the conditional probability of state $H$ exceeds .5 (it is in fact $21/30 = .7$) in this case. Thus if $B_1$ is 1, $B_2$ is 1 regardless of $S_2$, and we are in a cascade.

From here on we use notation like $[1, 0, 0, 0]$ for sequences of $S_t$'s (which are also $B_t$'s in the non-cascade sequences). With $[0, 1]$ as data at $t = 2$, the evidence is just as in the $[1, 0]$ case, so again we get $B_2 = 1$, but with $[0, 0]$ the conditional probability of $L$ is $.81/(.81 + .09) = .9$, so we get $B_2 = 0$. Thus an initial $S_1 = 0$ does not lead immediately to a cascade. The probability of $[0, 0, 1]$ under $H$ is .063, while under $L$ it is .081, and therefore $B_3 = 0$ after two initial 0's, regardless of the signal, and we see that $[0, 0]$ starts a cascade of sells. On the other hand, $[0, 1, 0]$ gives the same conditional probabilities as $[0, 0, 1]$ and thus leads to $B_3 = 0$, while $[0, 1, 1]$ leads to

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$B_3 = 1$. So $[0,1]$ does not automatically start a cascade. $[0,1,1,1]$ and $[0,1,1,0]$ both have higher probability under $H$, so $[0,1,1]$ starts a buy cascade. After $[0,1,0]$, we do not get a cascade unless two subsequent zeros occur. We do get both $[0,1,0,0,0,0]$ and $[0,1,0,0,0,1]$ leading to $B_6 = 0$, setting off a cascade.

So the overall probability of a cascade starting with the first observation is the probability of $S_1 = 1$, plus the probability of $[0,0]$, which is .79 under $H$ and .91 under $L$. The probability of $B_3$ being in a cascade is the probability of these first two events, plus the probability of $[0,1,1]$ plus the probability of $[0,1,0,0,0]$, which is .94276 under $H$ and .9846 under $L$.

An immediate false buy cascade occurs with an initial $S_1 = 1$ under $L$, which has probability under $L$ of .1. An immediate false sell cascade occurs with an initial $[0,0]$ under $H$, which has probability under $H$ of .09.

(2) Consider a model with two types of agent who can borrow and lend and trade a risky asset. Each has one unit of the risky asset at the start. The "borrowing and lending" is very simple — people simply trade promises to deliver one unit of consumption next period. The price of such a promise is normalized to one. The price of the risky asset is $Q$, meaning that one unit of the risky asset can be traded for $Q$ riskless promises of consumption next period. The risky asset will deliver two units of the consumption good if the state is $H$, one unit of consumption if the state is $L$. The two agent types are numbered $i = 1,2$. Agent $i$ believes the probability of the high state is $p_i$ and has a utility function $U_i$. Agent $i$ has consumption $C_{iH}$ in the high state and $C_{iL}$ in the low state. Formally, the problem of agent $i$ is

$$\max_{S_i,B_i}\{p_i U_i(C_{iH}) + (1-p_i)U_i(C_{iL})\} \quad \text{subject to}$$

$$QS_i + B_i = Q$$

$$C_{iH} = B_i + 2S_i$$

$$C_{iL} = B_i + S_i$$

Market clearing requires $B_1 + B_2 = 0$, $S_1 + S_2 = 2$. Note that the individual agents take the price $Q$ as given in doing their maximizations, and they do not see the market clearing constraints as applying to them individually. Consider two cases:

**pure risk sharing:** $p_1 = p_2 = .5$, $U_1(x) = \log(x)$, $U_2(x) = -1/x$

**pure difference of opinion:** $p_1 = .7$, $p_2 = .4$, $U_1(x) = U_2(x) = \log(x)$

(a) Find the first-order conditions for the two types of agents and use them (and some or all of the other equations given above) to construct a system of equations whose solution is the market equilibrium, determining $C_{iH}$, $C_{iL}$, $Q$, $S_i$, $B_i$, for $i = 1,2$. This is an application of Lagrange multiplier calculus methods, so your equations may involve Lagrange multipliers as well as the variables already appearing in the model.

The first order conditions are

$$\partial C_{iH}: \qquad p_i U_i'(C_{iH}) = \lambda_{iH}$$

$$\partial C_{iL}: \qquad (1-p_i)U_i'(C_{iL}) = \lambda_{iL}$$

$$\partial S_i : \qquad Q\mu_i = 2\lambda_{iH} + \lambda_{iL}$$
$$\partial B_i : \qquad \mu_i = \lambda_{iH} + \lambda_{iL}$$

(b) Solve the model for the two cases. This will be much easier than it might appear, because the answer is multiple-choice. Listed below are some possible values for $\{C_{iH}, C_{iL}, i = 1, 2\}$. You need only determine which two are answers for the two cases, then use them and the equations to find values for the remaining variables for those cases. (The numbers given will of course be solutions only up to the numerical precision permitted by the three decimal places shown.)

| | $C_{1H}$ | $C_{2H}$ | $C_{1L}$ | $C_{2L}$ |
|---|---|---|---|---|
| A | 2.545 | 1.455 | 0.667 | 1.333 |
| B | 2.243 | 1.757 | 0.817 | 1.183 |
| C | 2.255 | 1.745 | 0.898 | 1.102 |
| D | 2.625 | 1.375 | 0.512 | 1.488 |

The simple answer is: C solves the pure risk-sharing case and A solves the difference of opinion case. Given the $C_{iH}, C_{iL}$ values, one can calculate $S_i$, $B_i$ and $Q$ from the budget constraints and market clearing conditions. Then one just has to check the first-order conditions. An R program and a spreadsheet that do this are on the course website. The solutions for asset holdings and $Q$ are

$$S_i = C_{iH} - C_{iL}$$
$$B_i = 2C_{iL} - C_{iH}$$
$$Q = B_i/(1 - S_i)\,.$$

With Lagrange multipliers eliminated, the FOC's to check are just

$$Q = \frac{2p_i U_i'(C_{iH}) + (1 - p_i)U_i'(C_{iL})}{p_i U_i'(C_{iH}) + (1 - p_i)U_i'(C_{iL})}\,.$$