Fiscal Theory of the Price Level

1. First-Order Linear Differential Equations

A first-order linear differential equation has the form

$$\dot{y}_t = a_t + b_t y_t \,. \tag{1}$$

If a and b are constants, every function y_t of t that satisfies (1) can be written as

$$y_t = A_0 e^{bt} - \frac{a}{b} \,, \tag{2}$$

where A_0 is constant. The value of A can be pinned down if we know the value of y_t at any single value of t. For example, if we know y_0, the solution is

$$y_t = \left(y_0 + \frac{a}{b} \right) e^{bt} - \frac{a}{b} \,. \tag{3}$$

If b is constant but a varies over time, then the general solution can be written in any of the following ways:

$$y_t = Ae^{bt} + \int_0^\infty e^{bs} a_{t-s} \, ds \,, \tag{4}$$

$$y_t = Ae^{bt} + \int_0^t e^{bs} a_{t-s} \, ds = Ae^{bt} + \int_0^t e^{t-s} a_s \, ds \,, \tag{5}$$

$$y_t = Ae^{bt} + \int_0^\infty e^{-bs} a_{t+s} \, ds \,, \tag{6}$$

where A is a constant that is again pinned down once we specify y_t for some particular t. The first two versions are called "backward" solutions, and the last is called a "forward" solution. So long as all the integrals involved converge (as would be true, e.g., if a_t were zero except over some time interval (t_0, t_1)), each of these versions of a general solution applies—any solution is a special case of each formula, with differing A's. There is a further generalization possible, allowing for time-varying b, but we will not need that form of the solution.

When b is positive and a does not tend to zero as $t \to \pm\infty$ but does remain bounded, the infinite backward solution (4) is undefined. Because in this case the integral in the forward solution (6) is bounded, the solution tends to explode exponentially at the rate b if and only if $A \neq 0$. This makes the forward solution usually the most useful one for the $b > 0$ case. For similar reasons, one of the backward solutions is usually the most useful one for the $b < 0$ case.

2. Exercise due 11/30

You know from lectures and the model laid out in the assigned sections of the paper "Fiscal Foundations of Price Stability in Open Economies" that a government policy of setting a constant nominal interest rate and a constant primary surplus, i.e. $r = \bar{r}$, $\tau = \bar{\tau}$, results in a uniquely determined price level. It sometimes seems, though that the fiscal response to increased deficits brought about by increased debt service (interest payment) costs is slow. Also, discussions of monetary policy sometimes suggest that the Fed is willing to lower interest rates when government debt is brought down and thinks it must raise them when government debt rises. So suppose that the policy rules take the following form:

$$r = \theta_0 + \theta_1 b \tag{7}$$

$$\tau = -\phi_0 + \phi_1 b + \phi_2 r \; . \tag{8}$$

Under what conditions on the parameters in these equations is the outcome a uniquely determined price level?