Answers to Takehome

1. (a) We will spend $X(0)+B$ on asset purchases, acquiring $(X(0)+B)/Q(0)$ units of the asset. At t these will be worth $Q(t)(X(0)+B)/Q(0)$ dollars, and we will then have to pay back our borrowings, leaving $Q(t)(X(0)+B)/Q(0)-B$ as the final amount. To express this using q's instead of Q's, we write it as

$$e^{q(t)-q(0)}(X(0)+B)-B\,. \tag{1}$$

(b) The probability that the expression in (1) exceeds \$1000 is

$$P\left[e^{q(t)-q(0)} > \frac{1000+B}{X(0)+B}\right] = P\left[q(t)-q(0) > \log\left(\frac{1000+B}{X(0)+B}\right)\right]\,. \tag{2}$$

But the far right-hand side of (2) tends to the log of 1 as $B \to \infty$, and the log of 1 is zero. So the whole expression converges to the probability that $q(t)-q(0) > 0$. Since $q(t)-q(0)$ is a $N(0, .04t)$ random variable, this limiting probability is .5. It cannot exceed .5, unless $X(0) > \$1000$, and since you were given that $X(0) = \$100$, you know that in this example it cannot exceed .5.

(c) The strategy makes the probability of not earning \$1000 just .9 at each stage. Because q is a martingale, each step is independent of all the others. The probability of not earning \$1000 n times in a row is $.9^n$, which goes to zero as $n \to \infty$, which finishes our argument.

(d) The crucial thing that makes the strategy infeasible is the fact that borrowing capacities are not infinite. LTCM was regarded as extremely levered when it had debt equal to 20 times its capital. If there is a limit of $B \leq 10,000$ (a 100 to 1 leverage ratio in terms of initial capital), then the probability that the strategy loses money is around 40% (calculated by simulation). The expected gain is positive, because with $q \sim N(0,\sigma^2)$, $E[e^q] = e^{\frac{\sigma^2}{2}} > 1$. But this is made up of a substantial probability of a gain of around \$1000, together with smaller but substantial probabilities of losses of several thousand dollars.

It is also relevant to the feasibility of the strategy that it may require trading arbitrarily quickly, as the time intervals get shorter. This is not in fact possible. However, this only shows that the strategy can't guarantee a positive return in a finite span of time. With a minimum trading interval, the strategy still can guarantee a positive return, but over a possibly long span of time.

The fact that the interest rate is zero in the example is not in itself a reason the strategy could not work. With a positive interest rate, the target earnings at each date t would be, instead of \$1000, $\$1000e^{rt}$, and the amount paid back would include interest, instead of being just B. The strategy can be adjusted to deliver a profit with probability one nonetheless.

2. The objective function, after substituting out the C's using the constraints and taking expectations, is

$$\begin{aligned}&\gamma \cdot (4 - QX - Q^*Z)\\ &\qquad + .9 \cdot (\gamma \cdot (X+Z)(1-\pi) + \gamma\pi X - \big((1-\pi)(X+Z)^2 + \pi X^2\big)\ .\end{aligned} \tag{3}$$

Taking derivatives with respect to Z and X, we get as first-order conditions

$$\partial X: \qquad -\gamma Q + .9\,(\gamma \cdot (1-\pi) + \gamma\pi - 2(1-\pi)(X+Z) - 2\pi X) = 0 \tag{4}$$

$$\partial Z: \qquad -\gamma Q^* + .9\,(\gamma \cdot (1-\pi) - 2(1-\pi)(X+Z)) = 0\,. \tag{5}$$

Note that if we subtract (5) from (4), we get

$$\gamma \cdot (Q - Q^*) = .9(\gamma\pi - 2\pi X)\ , \tag{6}$$

which can be solved to produce

$$Q - Q^* = .9\pi \cdot \left(1 - \frac{2X}{\gamma}\right)\ . \tag{7}$$

Using the $X = 1$, $Z = 1$ equilibrium conditions and the given values for π and γ in (4) and (5) lets us solve for Q and Q^*, arriving at $Q = .6761$ and $Q^* = .6683$. This makes the difference in discount factors .0078, a bit smaller than the probability of default. If instead we look at $1/Q^* - 1/Q$, we get .0173, a bit bigger than the default probability. But in general, as can be seen from (7), the difference in yields depends on γ as well as the default probability. We can see, though, that as $\gamma \to \infty$, (7) implies that the difference in asset prices converges to $.9\pi$, while the difference in gross interest rates $1/Q^* - 1/Q$ converges to $.9^{-1}\pi/(1-\pi)$.

So we have completed the answer to both parts of the question.