Lending to Trade Risk

In this exercise we examine an extremely simple economy in which there are two kinds of people with different degrees of aversion to risk. We will see how, if they are allowed to borrow and lend, they trade risk to make themselves better off. We will also see that the gains from this risk-trading are modest, though the amount of trading is not.

The model is simple to describe, but not so easy to solve. There are two kinds of people, types 1 and 2, indexed by i. Each has a utility function over consumption of the form

$$E\left[C_i - \frac{\gamma_i}{2}C_i^2\right] . \tag{1}$$

The parameter γ_i measures the curvature in the utility function, and is therefore also a measure of the degree of *risk aversion* (see Figure 1). We assume that agents of type i have $\gamma = i$, so both agents are risk-averse, but type 2's are more risk-averse.

Each agent starts out with one unit of the same kind of asset. The asset delivers a random payoff z with mean $\mu = .1$ and variance $\sigma^2 = .01$. When they trade, they trade portions of the asset for promises to deliver the consumption good without uncertainty – i.e. risk-free assets. There is no existing asset in the economy that delivers a risk-free return, so these assets are "paper" assets, created as promises by one person to make a payment to another person. The price of the real, random-return asset is Q, representing the rate of exchange between one unit of this real asset and a promise to deliver one unit of the consumption good next period without risk. We let x_i stand for the amount of her initial holdings that a type i agent decides to sell. This amount

FIGURE 1. Measuring Risk Aversion

A person who maximizes the expectation of a linear utility function is said to be *risk-neutral*, because such a person is indifferent between being offered a random amount of the consumption good and being offered the expected value of the random amount as a certainty. That is, if utility is $U(C) = a+bC$, expected utility is $E[U(C)] = a+bE[C]$, so that the person's welfare is unaffected by changes in the variance of C. If utility is instead quadratic, so $U(C) = a + bC + cC^2$, expected utility becomes $a + bE[C] + cE[C]^2 + c\,\mathrm{Var}(C)$. In this case, with $E[C]$ held fixed, welfare changes with $\mathrm{Var}(C)$. Usually we assume that most people are *risk-averse*, which corresponds to $c < 0$ with quadratic utility, meaning that, for a given $E[C]$, they prefer smaller $\mathrm{Var}(C)$. However, there is nothing inherently irrational about $c > 0$, which implies that the person is *risk-loving*.

can turn out to be negative, which indicates that she decides instead to buy more of the risky asset than she started with. Because the only things being traded are these assets, every sale of the risky asset corresponds to a purchase of the risk-free asset, and vice versa. That means we must have for each agent

$$Qx_i = B_i \,, \tag{2}$$

where B_i is the amount of promised risk-free consumption that is purchased by the sale of the x_i units of the risky asset. Furthermore, in order for markets to clear, we must have

$$x_1 = -x_2 \tag{3}$$

(assuming there are equal numbers of the two types of agent).

The assets we have described are the entire income of the agents, so that

$$C_i = (1 - x_i)z + B_i \,. \tag{4}$$

That is, consumption of the i'th type of agent is the random return z on the amount of risky asset she retains, plus the amount of risk-free consumption she has purchased by giving up risky assets.

a. Substitute (4) into (1) and use (2) to obtain an expression for expected utility as a function of μ and σ^2, (the mean and variance of z), Q, γ_i and x_i.
b. Find the numerical values of utility for the two types of agent when they are not allowed to trade (i.e., $x_1 = x_2 = 0$).
c. Find an expression for optimal x_i as a function of Q and γ_i when the agents are allowed to trade. This is most easily done by differentiating w.r.t. x_i the expression for expected utility you found in (a) and setting the resulting expression to zero.
d. Find the equilibrium value of Q by using the market-clearing condition (3).
e. Find the value of x_1. Verify that it implies that the more risk averse agent sheds risk via asset trades, while the less risk averse agent takes on additional risk.
f. Compare the expected utility levels of the two agents in this equilibrium with trade to the levels in the no-trade case.