Econ. 487a Fall 1998 C.Sims

Martingale Examples

1. A Martingale with Stationary Increments Can't Have a Derivative

Suppose $\{X_t\}$ is a martingale whose increments all have finite variance. That is,

$$\mathrm{Var}\left(X_{t+s} - X_t\right) = E\left[(X_{t+s} - X_t) \cdot (X_{t+s} - X_t)'\right] = \sigma^2(s) < \infty \,. \tag{1}$$

In writing σ^2 in (1) as a function of s alone, not t, we are assuming further that the $\{X_{t+s} - X_t, t \in \mathbb{R}\}$ process is **stationary**, i.e. its properties do not depend on its absolute location in time, t. Note the following property of martingale increments:

Proposition 1. *If (a, b) and (c, d) are non-overlapping intervals and X is a martingale, then*

$$E[(X_b - X_a) \cdot (X_d - X_a)'] = 0\,.$$

Proof. Suppose (without loss of generality) that $a < b < c < d$. Then by the law of iterated expectations,

$$E[(X_b - X_a) \cdot (X_d - X_c)'] = E\left[(X_b - X_a) \cdot E_c[(X_d - X_c)']\right]\,.$$

But by the definition of a martingale, the right-hand side in this equation is zero. □

Now consider the interval $(a, a + b)$ and suppose that we divide it into n pieces, each of length b/n. Then we can write

$$\begin{aligned}\mathrm{Var}\left(X_{a+b} - X_a\right) &= \mathrm{Var}\left(\sum_{j=1}^{n} X(a + bj/n) - X(a + b\cdot(j-1)/n)\right)\\ &= \sum_{j=1}^{n} \mathrm{Var}\left(X(a + bj/n) - X(a + b\cdot(j-1)/n)\right) = n\sigma^2(b/n)\,. \end{aligned} \tag{2}$$

Since the left-hand end of (2) is constant as we change n, the right-hand end tells us that $\sigma^2(b/n)$ is proportional to b/n. We will denote the constant of proportionality by σ^2, so we can rewrite (2) as

$$\mathrm{Var}\left(X_{a+b} - X_a\right) = \sigma^2 b\,. \tag{3}$$

Suppose now that there were a time-derivative of X, that is another stochastic process, $\left\{\dot{X}(t)\right\}$ such that for every t, in some sense

$$\dot{X}_t = \lim_{\delta\to 0} \frac{X_{t+\delta} - X_t}{\delta}\,. \tag{4}$$

We need the "in some sense" here because we are taking limits of random variables, not of ordinary numbers. Until we are more specific, it is not clear what the "lim" in (4) means. There are several forms of convergence for limits of random variables. The one we will use here is called **mean square or quadratic mean** convergence. It is written

$$X_\delta \xrightarrow[\delta\to 0]{\text{q.m.}} Y \overset{def}{\Longleftrightarrow} E[(X_\delta - Y)^2] \xrightarrow[\delta\to 0]{} 0\,. \tag{5}$$

With this definition of convergence, we assume that $\dot{X}$ exists and is stationary with finite variance

$$\mathrm{Var}\left(\dot{X}_t\right) = \nu^2\,. \tag{6}$$

It is a property of q.m. convergence (which we won't prove here) that

Proposition 2. *If $Var(Y) < \infty$ and*

$$Z_\delta \xrightarrow[\delta\to 0]{q.m.} Y\,,$$

then

$$Var(Z_\delta) \xrightarrow[\delta\to 0]{} Var(Y)\,.$$

Proposition 2, together with our assumptions on the $\dot{X}$ process, imply that

$$\mathrm{Var}\left(\frac{X_{t+\delta}}{\delta}\right) \xrightarrow[\delta\to 0]{} \mathrm{Var}\left(\dot{X}_t\right) = \nu^2\,. \tag{7}$$

Yet we also know, from (3), that

$$\mathrm{Var}\left(\frac{X_{t+\delta}}{\delta}\right) = \delta^{-2}\,\mathrm{Var}\left(X_{t+\delta}\right) = \frac{\sigma^2}{\delta}\,. \tag{8}$$

The right-hand side of (8) converges to infinity as $\delta \to 0$, contradicting (7). So by assuming that a finite-variance $\dot{X}$ process exists, we have reached a contradiction.

2. A Poisson Martingale

A Poisson process is constant most of the time, but every once in a while jumps. If X_t is Poisson with parameter λ, the probability that there will be at least one jump in any time interval of length v is $1 - e^{-\lambda v}$. This turns out to imply that at any time t the conditional probability distribution of the time until the next jump, given information at t, has pdf $\lambda e^{-\lambda v}$. In the simplest version of this process, the size of the jump is always either $+1$ or -1, with equal probabilities. It is easy to see that this process is a martingale and that its time paths will look like a step function.

An interesting variant on this process is one in which the time-derivative of the path of X exists and is a positive constant, say θ, for all t at which there is no jump, while the jump itself is still $+1$ or -1, but now with the probability of -1 equal to $\pi > .5$. If π, θ and λ satisfy

$$\theta + \lambda\cdot(1 - 2\pi) = 0\,, \tag{9}$$

then this process is a martingale.

These examples are important, because they show that smooth behavior over any particular finite span of time is not inconsistent with a process being a martingale, and further that a process can show trending behavior over any particular finite span of time while still being a martingale. It is this kind of example that people have in mind when they suggest that an asset price time series might show a "peso problem". If there is a small probability of a large, discontinuous, drop in an asset price, then over a long span of time it could deliver above-normal returns, without this indicating any excess *expected* return. Investors would be factoring in the risk of a large price drop in evaluating overall expected returns.