Further Discussion of Complete vs. Incomplete Markets

An economy has complete markets if every kind of *state-date-contingent commodity* can, directly or indirectly, be traded in competitive markets. So if I want to trade umbrellas, delivered Thursday at 9AM if it doesn't rain, for bottles of beer, delivered Saturday at 8PM if the day has been warm and sunny and Yale won the football game, I can find a market relative price and trade these commodities for each other or for cash delivered now. This implies that the ratio of marginal utilities of different state-date-contingent commodities are the same for everyone. This is not a mysterious result. It is the same result as the one that in a competitive equilibrium the ratio of the marginal utility of bread to the marginal utility of cheese is the same for everyone. Or to put it another way, it is the same as the result that the slope of everyone's indifference curve at equilibrium is the same. It arises because utility maximizers will make indifference curve slopes match price ratios.

All you really need to understand for this course is the intuition in the paragraph above and the conclusion in (4) and (6) below. To give a more careful argument, as we are about to do, does not require fancy mathematics, but it does lead to some messy notation. You can do the November 2 exercise without following this argument in detail. You need only apply (6).

We consider a particularly simple economy, in which there is just one commodity. The amount of it that agent j consumes at time t is $C_j(t)$. This is a random variable, because the economy is affected by a random variable $X(t)$. To keep the mathematics simple we assume that $X(t)$ takes on only finitely many possible values at each t and that there is a finite upper limit T to t, though the results actually go through also for continuously distributed random variables and for infinite time horizons. The choices of individuals at t depend only on the history of X up to t, which we label $\mathcal{I}_t(X) = \{X(s) | s \leq t\}$. Suppose the objective of agent j is to maximize

$$E\left[U_j\left(\{C_j(t) | t = 0, \ldots, T\}\right)\right] = \sum_X \pi(X) U_j\left(\{C_j(t; \mathcal{I}_t(X)) | t = 0, \ldots, T\}\right) , \qquad (1)$$

where $\pi(X)$ is the probability of the sequence $X = \{X(s) | s = 0, \ldots, T\}$. With complete markets, each possible $C_j(t; \mathcal{I}_t(X))$ is an amount of consumption of a distinct commodity and has a price $P(t; X)$. So the budget constraint is

$$\sum_X \sum_{t=0}^{T} P(t; X) C_j\left(t; \mathcal{I}_t(X)\right) = \ldots \qquad (2)$$

The right-hand side of the equation is the value of the resources available to the individual. We don't need to know its nature, except that it does not depend on the individual's C choices. Applying Lagrange multipliers and taking the first-order condition with respect to $C_j(t;\mathcal{I}_t)$, we get

$$\sum_{X\,|\,\{X(1),\dots,X(t)\}=\mathcal{I}_t} \pi(X)\frac{\partial U_j}{\partial C_j(t;\mathcal{I}_t)} = \lambda_j \cdot \sum_{X\,|\,\{X(1),\dots,X(t)\}=\mathcal{I}_t} P(t;X)\,. \tag{3}$$

The partial derivative on the left-hand side of (3) does not vary with X, so long as we vary X without changing $\mathcal{I}_t(X)$—i.e. so long as we vary only the values of $X(s)$ for $s > t$.

If we now take the ratio of (3) for two individuals j and k, but for the same t, we emerge with

$$\frac{\partial U_j}{\partial C_j(t,\mathcal{I}_t)} \Bigg/ \frac{\partial U_k}{\partial C_k(t,\mathcal{I}_t)} = \frac{\lambda_j}{\lambda_k}\,. \tag{4}$$

A special case that occurs often in macroeconomic models is that of an *additively time separable* utility function, which means

$$U_j\left(\{C_j(t)|\,t=0,\dots,T\}\right) = \sum_{t=0}^{T} \Phi_t u_j(C_j(t))\,. \tag{5}$$

This gives us the simpler version of (4)

$$\frac{u_j'\left(C_j(t,\mathcal{I}_t)\right)}{u_k'\left(C_j(t,\mathcal{I}_t)\right)} = \frac{\lambda_j}{\lambda_k} \tag{6}$$

In (4) the left-hand side depends in principle on the whole time sequence of C's, so it is not in this form something we could check by observing the economy without knowing more about U. But in (6) we do get an implication we could check by observation. Because the form of u_j does not change in time, if we assume that $u_j'' < 0$ for all j, then over time C_j and C_k should move together.

In the exercise due 11/2, the utility function you are given is time separable and does not depend on j. So you can check whether in equilibrium (6) is satisfied.