1. Answers to Problem Set Due 9/14

a. We need to show that $E_t[P_{t+s}] = P_t$ for all integer $s > 0$. We are given that for all t, $E_t P_{t+1} = P_t$. We know

$$E_t[P_{t+s}] = E_t[E_{t+s-1}[P_{t+s}]] = E_t[P_{t+s-1}] \,, \tag{1}$$

with the first equality following from the law of iterated expectations and the second equality being just an application of the condition we are given on one-step-ahead expectation. But having done this once, we can repeat it to get $E_t[P_{t+s}] = E_t[P_{t+s-2}]$, etc. s times until we arrive at our target condition $E_t[P_{t+s}] = E_t[P_t] = P_t$.

b. At time 1, $P_1 = 2$ and there are just two possible values for P_3, 1 and 3. The one path for P that has $P_2 = 3$ has probability .5, and the sum of the probabilities of the other three paths is also .5, so

$$E_1 P_2 = .5 \cdot 3 + .5 \cdot 1 = 2 = P_1 \,.$$

At time 2, we may have P_2=3, in which case we are sure that $P_3 = P_4 = 3 = P_2$, so that the martingale property is trivially satisfied. If instead P_2=1, we can have either $P_3 = 2$ or $P_3 = 0$. There are two paths that make $P_3 = 2$, with total probability .25, and one path with $P_3 = 0$, which also has probability .25. The *conditional* probabilities of $P_3 = 2$ and $P_3 = 0$ given $P_2 = 1$ are therefore both .5, and we therefore have $E[P_3 | P_2 = 2] = 2$, satisfying the martingale property. Finally, P_3 can be 0, 2 or 3. When it is 0 or 3, the conditional probability of $P_3 = P_4$ is 1, so the martingale property is trivially satisfied. When $P_3 = 2$, we are on one of two paths, each of which has probability .125. Therefore the conditional probabilities of the two paths, given $P_3 = 2$, are both .5, and again we conclude $E[P_4 | P_3 = 2] = 2$, which validates the martingale property. We have now verified that $E_t P_{t+1} = P_t$, for each possible $t = 1, 2, 3$ and for each possible time path of P up to time t (i.e. for each possible point in the information set at each t). We have not checked $E_t[P_{t+s}] = P_t$ for $s > 1$, but (a) tells us this is unnecessary.

If, as proposed in the latter part of the exercise, we make $\pi_3 = .25$, $\pi_4 = .125$, the process is no longer a martingale. We still have $E_1 P_2 = P_1$ in this case, so there is no profit opportunity at $t = 1$. There is also no profit opportunity at $t = 2$ if $P_2 = 3$, as in that case we know with certainty that P_t will remain stuck at 3. But if $P_2 = 1$, the probability of paths with $P_3 = 2$ is now $.125 + .25 = .375$ and that of the path with $P_3 = 0$ is .125, making the conditional probabilities .75 and .25. Thus

$$E[P_3 | P_2 = 1] = .75 \cdot 2 + .25 \cdot 0 = 1.5 > 1 \,.$$

The profit opportunity therefore arises at $t = 2$, in the case where $P_2 = 1$. The expected yield on an investment made in these circumstances is 50%. At $t = 3$ the deviation from martingale behavior occurs only for $P_3 = 2$. In this case, the conditional expected return is negative ($-16\frac{2}{3}\%$). The profit opportunity is therefore obtained here by selling the asset at time 3, rather than buying it, in case $P_3 = 2$.

c. We will apply the formula from the notes

$$E[X_2|X_1] = \mu_2 + (X_1 - \mu_1)'\Sigma_{11}^{-1}\Sigma_{12} \,. \tag{2}$$

In each part of this problem, we are given a 3×3 covariance matrix we will write as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \,. \tag{3}$$

In forming E_1P_2, we have P_2 corresponding to X_2 in (2), P_1 corresponding to X_1, σ_{11} to Σ_{11}, and σ_{12} to Σ_{12}. All these terms are 1×1, or what is sometimes called **scalar**, meaning non-matrix. In forming E_2P_3 things get slightly more complicated, with the role of X_1, the conditioning information set, being played now by P_1 and P_2. The correspondences for this case can be listed as

$$\begin{aligned} X_1 &\to \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ X_2 &\to P_3 \\ \Sigma_{11} &\to \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \\ \Sigma_{12} &\to \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix} \end{aligned}$$

For the first covariance matrix (a), there was a mistake in the problem statement. The matrix given cannot be a covariance matrix. (It would imply that the variance of $P_2 - P_1$ is negative, which is impossible, since a variance is the expectation of a square, which is always non-negative.) If we nonetheless went ahead and applied the formula, it would give us

$$E_1P_2 = 2 + (P_1 - 2)\frac{1}{1} \cdot 2 = 2P_1 - 2 \neq P_1 \,,$$

which does not satisfy the martingale property. It would also give us

$$E_2P_3 = 2 + [P_1 - 2 \;\; P_2 - 2] \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = .5P_2 + 1 \neq P_2 \,,$$

again violating the martingale condition.

Similar calculations show that (b) is a martingale and (c) and (d) are not. In the case of (d), $E_1P_2 = P_2$, but $E_2P_3 \neq P_2$.

2. Revised version of Problem 3 if you're handing it in a week late

Don't bother with covariance matrix (a), which was a mistake and is worked out above. Show the arithmetic for (b), (c) and (d). *Or* prove that in the covariance matrix for a martingale, where the typical element is σij,

$$\sigma_{ij} = \sigma_{ii} \text{ for all } j \geq i \,, \tag{4}$$

then use this fact that you've proved to get the right answers for (b), (c), and (d) without doing any arithmetic. To prove (4), use the facts that

$$\sigma_{ij} = E[(P_i - \mu_i) \cdot (P_j - \mu_j)] \tag{5}$$

and

$$P_j = P_i + (P_j - P_i) \,. \tag{6}$$

The argument uses the same idea as the argument in class that changes of martingales over non-overlapping time intervals have zero covariance. Note also that for a martingale P_t it must be true that the unconditional mean $E[P_t]$ is constant, not dependent on t.