TAKE-HOME EXAM

(1) Consider the problem of a firm that needs to track with its own price both its own costs c and its competitors' average price p^*. It also faces an information-processing cost θ per nat of observing the randomly varying (c, p^*) vector. Its own costs are correlated with its competitors' prices. We approximate this problem with a linear-quadratic optimization problem with a Shannon information cost:

$$\max E_p[-\tfrac{1}{2}(p-p^*)^2 - \tfrac{1}{2}(p-c)^2] - \theta\big(\log|\Sigma| - \log|\Omega|\big) \tag{1}$$

$$\text{subject to} \quad \Sigma \succcurlyeq \Omega\,, \tag{2}$$

where Σ is the covariance matrix of p^* and c before collecting information and Ω is their covariance matrix after receiving the information. $\Sigma \succcurlyeq \Omega$ means "$\Sigma - \Omega$ is positive semi-definite".

(a) Show that in this problem the "no-forgetting" constraint $\Sigma \succcurlyeq \Omega$ always binds, because only one dimension of variation in (p^*, c) matters.

(b) Explain how to calculate the optimal Ω from given values of Σ and θ. Your answer can be a computer program, or a detailed description of the calculation that a programmer could use to write such a program.

Without uncertainty or information collection, the solution would be to set

$$p = \pi = \frac{p^* + c}{2}\,.$$

Applying certainty equivalence, therefore, the solution subject to uncertainty sets

$$p = \hat{\pi} = E[\pi \mid p]\,,$$

where we are using the fact that conditioning on the choice variable p is equivalent to conditioning on the full post-information-collection information set. (Since collection of information unrelated to the decision p would be wasteful.)

Now the objective function for the problem can be rewritten as

$$E[-(p-\pi)^2 - \tfrac{1}{2}(p^*-c)^2] - \theta(\log|\Sigma| - \log|\Omega|)$$
$$= E[-(p-\pi)^2] - \tfrac{1}{4}[1 \;\; -1]\Sigma\begin{bmatrix}1\\-1\end{bmatrix} - \theta(\log|\Sigma| - \log|\Omega|)\,.$$

The problem has become one-dimensional. Since the optimal choice of p depends on the p^*, c vector only through π, the solution will reduce uncertainty only about π. The problem can be reduced to maximizing

$$-\operatorname{Var}(\pi \mid p) - \theta(\log \operatorname{Var}(\pi) - \log(\operatorname{Var}(\pi \mid p))\,.$$

The solution is then just $\operatorname{Var}(\pi \mid p) = \theta$, so long as $\theta < \operatorname{Var}(\pi)$. $\operatorname{Var}(\pi)$ is one fourth of the sum of the elements of Σ. The uncertainty about π, and the losses, increase

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with information cost θ up to the point where θ matches the unconditional variance of π, at which point the optimal solution is to collect no information and set p equal to the unconditional expectation of π.

To compute the optimal Ω it helps to transform (p^*, c), to orthogonal variables:

$$\begin{bmatrix} \pi \\ z \end{bmatrix} = A \begin{bmatrix} p^* \\ c \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \sigma_{22} + \sigma_{12} & -\sigma_{11} - \sigma_{21} \end{bmatrix}$$

$$\text{Var}\left(\begin{bmatrix} \pi \\ z \end{bmatrix} \right) = A \Sigma A' = \begin{bmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix} .$$

So long as $\theta < \text{Var}(\pi)$, the solution changes the covariance matrix of (π, z) by reducing its upper left element from σ_π^2 to θ, while leaving σ_z^2 unchanged. So in the original coordinates we have

$$\Omega = A^{-1} \begin{bmatrix} \theta & 0 \\ 0 & \sigma_z^2 \end{bmatrix} (A^{-1})' .$$

Note that this means

$$\Sigma - \Omega = A^{-1} \begin{bmatrix} \sigma_\pi^2 - \theta & 0 \\ 0 & 0 \end{bmatrix} (A^{-1})' ,$$

which is of rank 1, thus positive semi-definite, not positive definite. If the no-forgetting constraint were not present, we could increase the objective function by increasing σ_z^2. (It would also increase $E[(p^* - c)^2 \mid p]$, but since the objective function is defined to depend on the unconditional expectation, the math of the problem leads to improving the objective function by increasing σ_z^2.)

(2) This is the standard flex-price FTPL model with long debt that you worked on in an exercise, except that now the output endowment Y is growing. Using A for the amount of consols the representative agent holds (and the government has issued), the private agent optimization problem is

$$\max_{C,A} \int_0^\infty e^{-\beta t} \log(C_t)\, dt \tag{3}$$

$$\text{subject to} \quad C + \frac{\dot{A}}{\rho P} + \tau \leq \frac{A}{P} + Y . \tag{4}$$

We assume $Y_t = Y_0 e^{gt}$, i.e. a constant growth rate for Y.

The government budget constraint is

$$\frac{\dot{A}}{\rho P} + \tau \geq \frac{A}{P} . \tag{5}$$

(a) With a policy that fixes τ and ρ at constant values, is there a unique equilibrium consistent with the private transversality condition for every possible value of g?

(b) For cases where equilibrium exists and delivers a unique initial price level, show how to determine the initial price level as a function of τ, g, ρ and the initial A_0.

As before, the FOC's are

$$\partial C: \qquad \frac{1}{C} = \lambda$$

$$\partial A: \qquad -\frac{d}{dt}\left(\frac{\lambda}{\rho P}\right) = \frac{-\dot{\lambda}}{\rho P} + \frac{\beta\lambda}{\rho P} + \frac{\lambda}{\rho P}\frac{\dot{P}}{P} + \frac{\lambda}{\rho P}\frac{\dot{\rho}}{\rho} = \frac{\lambda}{P}.$$

Using the constancy of ρ, social resource constraint $Y = C$ (derivable from the GBC and the private budget constraint) the FOC's can be reduced to

$$g + \beta + \frac{\dot{P}}{P} = \rho\,,$$

which determines the inflation rate $\dot{P}/P$.

The private TVC is

$$e^{-\beta t}\frac{\lambda A}{\rho P} = \lambda_0 e^{-(\beta+g)t}\frac{A}{\rho P} \to 0$$

But the GBC can be written as

$$\frac{d}{dt}\left(\frac{A}{\rho P}\right) = (\rho - (\rho - \beta - g))\frac{A}{\rho P} - \tau = (\beta + g)\frac{A}{\rho P} - \tau\,.$$

It is easy to see that, so long as $\beta + g > 0$, this is an unstable differential equation, and it has a unique stable solution because of the TVC. The unique stable solution is

$$\frac{A}{\rho P} = \frac{\tau}{\beta + g}\,. \qquad (*)$$

Private agents discount the future more heavily when g is positive, because future C will be higher, and therefore future marginal utility lower.

With ρ fixed and A_0 inherited from history, there is only one value of P_0 that aligns the real value of debt with the value determined by $(*)$,

$$P_0 = \frac{A_0(\beta + g)}{\tau\rho}\,.$$

Of course g can be negative. The analysis above still goes through so long as $\beta + g > 0$. But if $g < -\beta$, the GBC is no longer an unstable equation. In fact in this case there is no equilibrium with positive, finite, government debt given the stated policies. The backward solution of the GBC applies, and with $\tau > 0$ this means debt hits its zero lower bound in finite time. In fact, marginal utility of consumption is increasing so fast that agents have unbounded demand for government debt. They would like to move consumption from the present to the future, even with the negative real return on debt, no matter what the initial debt is, but the technology (with no real capital) implies that is impossible.

(3) Make an argument that in a period like the present, when interest rates on government debt and inflation are very low, it makes sense for the government to finance productive government investment by debt issuance. Also make the case that this does not make sense. Support both arguments with references to the literature.