ANSWERS TO SIMS 521 FINAL QUESTIONS

(3) [45 points]

Consider an economy in which the only way for agents to save is in the form of government nominal one-period bonds or interest-bearing money. The representative agent solves

$$\max_{\{C_s,B_s,M_s\}} E\left[\sum_{t=1}^{\infty} \log(C_t)\beta^t\right], \quad \text{subject to} \tag{1}$$

$$C_t \cdot (1+\gamma v_t) + \frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + Q_{t-1}M_{t-1}}{P_t} + Y_t - \tau_t\,, \quad \text{all } t \geq 0\,. \tag{2}$$

C_t is consumption at t, B_t is bonds held at t, R_t is gross nominal interest rate on bonds issued at t, Q_t is the gross nominal interest rate on money issued at t, τ_t is a lump-sum tax, and Y_t is an exogenous endowment. v_t is velocity of money, defined as $v_t = P_tC_t/M_t$. Assume that Y_t is i.i.d., bounded away from zero, and exogenously given.

The government has the budget constraint

$$\frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + Q_{t-1}M_{t-1}}{P_t} - \tau_t\,. \tag{3}$$

Suppose monetary policy consists of fixing arbitrarily $R_t \equiv \bar{R} > \bar{Q} \equiv Q_t$, i.e. pegging both interest rates.

FOC's of the agent are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t(1+2\gamma v_t)$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = \beta R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right]$$

$$\partial M: \qquad \frac{\lambda_t}{P_t}(1-\gamma v_t^2) = \beta Q_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right].$$

Taking the ratio of the B and M FOC's gives us

$$(1-\gamma v_t^2) = \frac{Q_t}{R_t}\,.$$

Thus velocity is determined uniquely by the policy of fixing R and Q, so long as $R > Q$, as the problem assumes, and so long as the policy is indeed consistent with equilibrium, which we still need to check. Using this result, the C FOC and the B FOC gives us

$$\frac{1}{P_tC_t} = \beta R E_t\left[\frac{1}{P_{t+1}C_{t+1}}\right].$$

Depending on whether $R\beta$ is bigger or less than one, this may be a stable or unstable difference equation in $1/PC$. With constant v, the social resource constraint (obtained by subtracting the government budget constraint from the private budget constraint, implies that C/Y is constant. Thus the C process is determined by the exogenous Y process and any implications we could draw for stability or instability of PC from this equation would reflect only properties of P. Transversality and feasibility arguments will not restrict P's asymptotic behavior, so we cannot use this equation to pin down an initial price level.

The government budget constraint in real terms is

$$\frac{B_t + M_t}{P_t} = R\frac{B_{t-1}}{P_t} + Q\frac{M_{t-1}}{P_t} - \tau_t\,.$$

The constancy of velocity implies that

$$\frac{M_t}{P_t} = \frac{C_t}{v} = \frac{Y_t}{v(1+\gamma v)}\,.$$

Dividing the GBC by C_t (dividing by Y_t would work also) and applying the E_{t-1} operator gives us

$$E_{t-1}\left[\frac{B_t}{P_t C_t}\right] + v = \beta^{-1}\frac{B_{t-1}}{P_{t-1}C_{t-1}} + \frac{Q}{R}\beta^{-1}v - \tau_t\,.$$

If fiscal policy sets τ_t so as to respond positively to B_t/P_tC_t, this equation could be stable and leave the system with no way to determine a unique initial price level. However if it responds weakly or not at all, the equation becomes unstable. The unique stable solution in the case of $\tau_t \equiv \bar{\tau}$ is

$$\bar{b} = \frac{B_t}{P_t C_t} \equiv \frac{v\left(1 - \frac{Q}{R}\beta^{-1}\right) + \bar{\tau}}{1-\beta}\,.$$

The instability requires that if $B_t/(P_tC_t)$ is not equal to its unique stable value, then it either becomes arbitrarily large with positive probability, or arbitrarily small. Here arbitrarily small means negative with arbitrarily large absolute value, as the equation is linear. Negative government debt is usually ruled out by assumption, though here the assumption is not explicit. Unbounded lending by the government to the public would actually be inconsistent with equilibrium, since then optimal private behavior by individuals would be arbitrarily large borrowing and arbitrarily large consumption, which is infeasible. If negative government debt is not allowed, then on a path consistent with the FOC's in which debt eventually becomes negative, agents see that consuming the equilibrium time path of C_t will lead them to deplete their wealth and eventually be forced to cut back on C_t. Foreseeing this, they will try to reduce C_t today, putting downward pressure on prices, thus raising the value of B/PC, until its unique equilibrium value is reached. If instead B/PC is on a path that requires it eventually grow at the rate β^{-t} (as implied by the GBC), the private transversality condition, here

$$\beta^t E\left[\frac{B_t}{P_t C_t}\right] \to 0\,,$$

is violated. Agents will feel richer than is consistent with equilibrium, they will try to increase consumption above its equilibrium level, thus raising prices and reducing real debt.

So the only solution consistent with equilibrium is the stable one. Using this fact in the original GBC (without the E_{t-1} operator but still divided through by $P_t C_t$) gives us

$$\bar{b} + v = \frac{RB_{t-1} + QM_{t-1}}{C_t P_t} - \frac{\bar{\tau}}{C_t} .$$

Everything in this equation is either fixed by policy or predetermined at time t, except P_t and C_t. Since C_t is determined by the exogenous Y_t, this equation determines P_t uniquely.

(a) Is this policy feasible?
Yes, we have shown it is.

(b) If so, are there restrictions on what fiscal policies are consistent with feasibility for this two-interest-rate-peg policy?
We need active fiscal policy to deliver a uniquely determined price level, but passive fiscal policies also allow equilibrium — just too many of them.

(c) Can it deliver a uniquely determined price level? If so, what restrictions, if any, are needed on fiscal policy?
We need active fiscal policy for this. To be able to use the transversality condition, we need that τ respond not at all, or inversely, to the level of real debt.

(d) How is the rate of inflation in this model related to the levels chosen for $\bar{R}$ and $\bar{Q}$?
$1/(P_t C_t)$ grows exponentially, so inflation is negative on average, when $\beta R < 1$, and the opposite is true when $\beta R > 1$. So higher R is associated with higher inflation.

(e) What effect does the spread $R - Q$ have on the nature of the equilibrium? On agent utility?
$Q/R = 1 - \gamma v^2$ in equilibrium, so a smaller gap between Q and R implies lower velocity, and thus lower transactions costs and higher agent utility.

(4) [20 points]

A vendor is trying to keep prices in line with fluctuating costs. The costs are i.i.d. and their values are always 10, 15, or 20. Ideally he would like to have a margin of five, pricing at 15, 20 and 25, respectively as the costs change, and profits suffer when he misprices. He has many products to price each day, so he faces an information constraint in tracking the individual costs. For one item, the distribution of costs among the three values is .3, .4, .3 as probabilities for 10, 15 and 20, respectively. The following table of joint probabilities for the costs and prices has been proposed as optimal:

price	cost		
	10	15	20
15	.2	.1	0
20	.1	.2	.1
25	0	.1	.2

Show that, if there is a cost to mutual information in Shannon's sense between product prices and product costs, this proposed solution cannot be correct.

Here the action is the price and the thing being tracked is the cost. The cheap way to answer this question is just to cite the result I gave in class that in a tracking problem $f(x, y) = 0$ only at points x such that $f(x, z) = 0$ for all z. However, this example did not tell you that this was a pure tracking problem, i.e. one with losses dependent only on cost - price. So a really good answer would have derived the result, as follows:

The mutual information is

$$\sum_{i,j} p_{ij} \log p_{ij} - \sum_{i,j} p_{ij} \log \left(\sum_k p_{ik} \right) - \sum_{i,j} p_{ij} \log \left(\sum_k p_{kj} \right) .$$

Since the marginal for costs is given, $\sum_k p_{kj}$ is fixed, and the last term in this sum can't be changed. Introducing the notation $q_{j|i} = p_{ij}/sum_k p_{ik}$ for the conditional density of price (j) given costs (i) and $\pi_i = \sum_j p_{ij}$ for the unconditional density of price, we can rewrite the mutual information as

$$\sum_{ij} q_{j|i} \log(q_{j|i}) \pi_i$$

plus a constant. That is, it is the expected entropy of j given i, minus the unconditional entropy of j, which is fixed. π and q completely parameterize the free dimensions of the choice problem, except that $q_{j|i}$ is required to sum to one in j. The derivative of mutual information with respect to $q_{j|i}$ is (ignoring the unit sum constraint) $1 + \log q_{j|i}$, which approaches $-\infty$ as $q_{j|i} \to 0$. Increasing $q_{j|i}$ slightly from zero, thus has arbitrarily large marginal effect in reducing mutual information. Of course, in increasing $q_{j|i}$ we would have to decrease $q_{j'|i}$ for some other value j' of costs. But so long as we chose this other value j' so that $q_{j'|i} > 0$, the marginal effect of reducing its value is finite and thus would not offset the infinite marginal effect of increasing $q_{j|i}$ from its zero value. This argument then implies that $q_{j|i} > 0$ for all j whenever the information constraint binds (so that decreasing it would matter) and at the same time $\pi_i > 0$. When $\pi_i = 0$, the conditional entropy of $j|i$ has no weight in the constraint, so manipulating it would have no effect.

The conclusion is that in any information-constrained problem with a decision variable x and a random variable y subject to information-costly observation, the solution will have $q_{y|x} > 0$ for all y with positive unconditional probability whenever the constraint binds and the marginal density on x is positive.

(5) [25 points] Answer two of the questions below.

(a) Is Brunnermeier and Parker's theory of "optimal expectations" just robust control with a sign change, or is there a more fundamental distinction?

There is a more fundamental distinction. B&P do postulate agents who distort probabilities in an optimistic direction, but instead of specifying a limited range of prior beliefs disciplined by distance from some central model, as in robust control, they allow any beliefs. They nonetheless end up predicting beliefs that are not too distorted, because they use an objective function that is the objectively expected time average expected utilities under the distorted beliefs. That is, at each date "felicity", instead of being a function of (say) current consumption, includes expected future utility under the distorted measure. Thus if the distorted beliefs lead to very bad decisions, objectively expected future "felicities" will be low. Their agents end up acting as Bayesians, but with a tendency to greater risk-taking than agents with undistorted beliefs, and in equilibrium their agents endogenously end up with heterogeneous beliefs. Robust control does not imply endogenously heterogeneous beliefs, though it does imply less risk taking than that by an agent with the central model's beliefs. One could imagine a model like those of Sargent and Hansen, restricting probabilities not to stray too far in Kullback-Leibler distance from a central model, but then choosing the probabilities to be the most optimistic rather than the most pessimistic in the set. But that is not the Brunnermeier-Parker model.

(b) Simple Phillips curve correlations of inflation and unemployment broke down in the 1970's. How does what goes on the left-hand-side in the Phillips curve regression interact with this breakdown in Primiceri's and in Sargent-Williams-Zha's learning stories about inflation policy in the 1970's?

If policy makers think of themselves as controlling output to influence inflation, as in an "inflation on the left" Phillips curve, then the breakdown of the inflation-output correlation in the 1970's implies that the output costs of bringing inflation down might be extremely high. Whereas if they think of themselves as controlling inflation to influence output, the breakdown of the correlation suggests that inflation could be brought down promptly with little or no effect on output. The former effect plays a big role in Primiceri's story. The latter effect probably plays a small role in the SWZ story, for two reasons. One is that their model as estimated makes it optimal for policy to have continued to increase inflation in the 1970's, despite the breakdown of the correlation. Another is that their implied estimates of the Phillips curve made policy-makers believe that unemployment would oscillate persistently unless they moved inflation to counteract it — in other words they have policy makers finding a more complicated pattern of empirical results than a simple "breakdown of the correlation".

(c) Are there testable implications of robust control decision theory that would allow us to distinguish it from ordinary Bayesian decision theory in the behavior of financial markets? Explain your answer.

It depends what version. If one applies robust control so as to make it equivalent to choosing a pessimistic measure at "$t = 0$", then acting as a Bayesian with that measure thereafter, there would be no testable distinction. Either type of agent would be acting according to the rules of Bayesian inference. A point I did not make in my lecture on this, which I am now 90% sure is correct (Hansen and Woodford both say it's true, and it looks plausible to me) is that the multiplier-preferences version of robust control does end up equivalent to choosing a pessimistic measure initially. The constraint version of robust control probably does not, though Hansen thinks maybe it does. In any case, it's clear that if the choice of pessimistic measure is redone each period, it is possible, depending on how the constraints or costs are formulated, for the robust control approach to lead to time-inconsistent behavior, which would be distinguishable from Bayesian optimization.

(d) Show that if investors have CRRA utility, returns are normally distributed, and investors have to estimate the mean and variance of returns from historical data, the usual Euler equations we use in asset valuation can easily become undefined.

Weitzman's way of showing this is to suppose that everyone agrees that yields are log-normally distributed, but that their logs have unknown mean and variance. (If absolute rather than log yields were normal, negative yields and, in equilibrium therefore, negative consumption would be possible), and if people update beliefs using "conjugate priors", a convenient and not unreasonable form of prior, then their distribution for returns conditional on the data will not be Normal, but instead Student-t. In a simple model where there is just one asset and in equilibrium consumption is equal to the total return on that asset, the Euler equation from differentiating with respect to holdings of that asset will be

$$Q_t U'(C_t) = \beta E_t[U'(C_{t+1})(Q_{t+1} + e^{z_{t+1}})] \,,$$

and therefore under usual transversality conditions

$$Q_t U'(C_t) = \sum_{s=1}^{\infty} \beta^s E_t[U'(C_{t+s}) e^{z_{t+s}}].$$

With CRRA utility, the expectation on the right is

$$\int_{-\infty}^{\infty} e^{(1-\gamma) z_{t+s}} f(z_{t+s}) \, dz_{t+s} \,.$$

If z_{t+s} is t-distributed conditional on information at t, then the tails of its density function f decline at a polynomial rate. Since z appears exponentiated in the integrand, the integrand does not converge. Unless $\gamma = 1$, the either the left tail or the right tail of the integral explodes. That is, with $\gamma > 1$, so agents are fairly risk-averse, the possible returns in low-consumption states are likely enough and valuable enought that the security becomes infinitely valuable as a hedge against those bad outcomes, while if $\gamma < 1$ so that agents have little risk aversion, the possible high returns are likely enough and valuable enough that the security becomes infinitely valuable.