(1) The New Keynesian/Taylor rule literature has worked almost entirely with the assumption that there is a single interest rate on government bonds, and that money (government liabilities valued for transactions services) pays no interest. The US now, following many other countries, pays interest on reserves (though there is still non-interest-bearing currency). In a very simple setting (ignoring currency) we explore what happens to the "Taylor principle" (policy should make interest rates increase more than proportionately with increased inflation) when there are two interest rates to be determined.

The model is that of "A Simple Model...", extended to include interest-bearing money. Individuals solve

$$\max_{C,B,M} \left[\sum_{t=0}^{\infty} \beta^t \log C_t \right] \quad \text{subject to} \tag{1}$$

$$C_t(1 + \gamma f(v_t)) + \frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + M_{t-1}S_{t-1}}{P_t} + Y_t - \tau_t \tag{2}$$

$$v_t = \frac{P_t C_t}{M_t} \tag{3}$$

$$B_t \geq 0 \qquad M_t \geq 0\,. \tag{4}$$

Y_t is assumed i.i.d. and bounded away from zero with probability one.

The government budget constraint is

$$\frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + S_{t-1}M_{t-1}}{P_t} - \tau_t\,. \tag{5}$$

Assume in what follows that $f(v_t) = v_t/(1 + v_t)$.

(a) Consider the policy of fixing at constant levels both S_t/R_t and M_t, with passive fiscal policy. Does this guarantee a unique equilibrium price level?

In-class discussion suggested people got this part, so I don't write out a detailed answer here.

(b) Fixing S_t/R_t is not feasible in reality if it is possible that R_t emerges so close to one that $S_t < 1$ is implied by the fixed ratio. In this model as it stands $S_t < 1$ is actually quite possible — the central bank just charges a fee for reserve deposits, rather than paying interest on them. But in the background is the presence of currency, which has a gross nominal rate of return of one and is freely exchangable with reserves. So we consider another possible policy: $S_t = \sqrt{R_t}$. With this policy, whenever R_t is greater than one, S_t lies between it and one. Can this policy, combined with a fixed-M policy and passive fiscal policy, deliver a unique equilibrium price level?

FOC's yield

$$\left(1 - \gamma \frac{v_t^2}{(1+v_t)^2}\right) = \frac{S_t}{R_t} = R_t^{-\frac{1}{2}}$$

$$z_t = \beta R_t E_t z_{t+1} \quad \text{with}$$

$$z_t = \frac{\bar{M}\lambda_t}{P_t} = \frac{1}{v_t(1 + \gamma v_t/(1+v_t) + \gamma v_t/(1+v_t)^2)} .$$

z_t is monotone decreasing in v_t, $z_t \to \infty$ as $v_t \to 0$ and $z_t \to 0$ as $v_t \to \infty$. (Check the monotonicity by just taking a derivative.) There is a solution with z_t, and hence v_t, constant, and with

$$\bar{R} = \left(\frac{1}{1 - \gamma \dfrac{v_t^2}{(1+v_t)^2)}} \right)^2 = \beta^{-1} ,$$

so long as $1 - \gamma < \sqrt{\beta}$. If $1 - \gamma > \sqrt{\beta}$, money's transactions value does not justify holding it and there is no equilibrium with valued money. We see this because in that case $E_t z_{t+1} > \theta z_t$ for a $\theta > 1$ on any path satisfying the FOC's, which implies that on such a path v is unbounded above. But the usual transversality argument makes such paths impossible.

If $\gamma > 1$, there is a unique stationary equilibrium. βR_t, the coefficient in front of $E_t z_{t+1}$ in (1b), is increasing in v_t, so if z_t goes above its steady-state value, meaning v decreases, R decreases. Thus for the FOC to hold, there must be non-zero probability of z_{t+1} exceeding z_t by a factor greater than one. Since this argument can be repeated at each subsequent date, and the factor rises at each date so long as v keeps rising, z must with non-zero probability eventually exceed any bound. But from (1b) we see that R becomes infinite at a finite value of v when $\gamma > 1$, and thus such paths are impossible.

This argument does not work, though, when $0 < 1 - \gamma < \sqrt{\beta}$. In that case R_t approaches the limit $1/(1-\gamma)^2$ as $v_t \to \infty$. z_t then shrinks steadily at the limiting rate $(1-\gamma)^2/\beta$, which implies v_t eventually grows at the inverse of that limiting rate. With M_t fixed and transactions costs approaching a fixed proportion γ of endowment, this means that P growth approaches the same limiting rate. Along such a path no equilibrium condition is violated and real balances converge to zero. Thus in this case equilibrium is not unique.

The paths in which v_t deviates from steady state and then shrinks without bound can be ruled out by a direct transversality argument, the same one given in class (and also in the underlying paper). This argument is needed to guarantee existence of equilibrium.

Thus the cases are:

(i) $\gamma > 1$: unique equilibrium
(ii) $1 - \beta < \gamma < 1$: multiple equilibia
(iii) $\gamma < 1 - \beta$: no equilibrium

The cases $\gamma = 1$ and $\gamma = 1 - \beta$ are trickier. $\gamma = 1$ means that $R_t \to \infty$ as $v_t \to \infty$, but in the model there is nothing to stop R from becoming arbitrarily large, so there are multiple equilibria. In the other multiple equilibria cases inflation converges to a steady state, while here it increases without bound.. $1 - \gamma = \sqrt{\beta}$ implies that the only steady state would be at $v = \infty$. Thus from any initial condition v would have to decrease, and there cannot be a lower bound on v. But we know by the usual transversality argument that paths with arbitrarilly small v are impossible. Therefore this is another case with no equilibrium with valued money.

(c) In any case above in which you found the initial price level indeterminate, could a unique equilibrium be restored by replacing passive with active fiscal policy, while keeping the monetary policy rules as specified?
The simplest active fiscal policy, $\tau_t \equiv \bar{\tau}$, does guarantee a unique equilibrium in the multiple-equilibria cases. It of course fixes the initial price level as the one that satisfies $(1-\beta)R_{-1}B_{-1}/\bar{\tau} = P_0$. This pins down the initial price level and thereby chooses one of the many paths that satisfy the FOC's.

(2) Here we use the Simple Model to explore how endogenous switching makes inflation sensitive to fiscal disturbances even during a period of apparent active money, passive fiscal, policy. All debt outstanding is consols (G — paying one unit of currency per time period for each consol and selling at nominal price $1/a_t$), but there is a well-defined short rate because agents see a possibility of lending or borrowing a short security.

Agents solve

$$\max_{C,B,G} E\left[\beta^t \log C_t\right] \quad \text{subject to} \tag{6}$$

$$C_t(1+\gamma f(v_t)) + \frac{B_t + G_t/a_t + M_t}{P_t} = Y_t + \frac{G_{t-1}}{P_t} + \frac{G_{t-1}}{P_t a_t} + \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} - \tau_t \tag{7}$$

$$B_t \geq 0 \quad G_t \geq 0 \tag{8}$$

The government budget constraint is

$$\frac{B_t + G_t/a_t + M_t}{P_t} = \frac{G_{t-1}}{P_t} + \frac{G_{t-1}}{P_t a_t} + \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} - \tau_t \tag{9}$$

Fiscal policy sets $B_t \equiv 0$ (but private agents do not see this as a constraint on themselves — they must freely choose $B_t = 0$ in equilibrium).

The policy on the primary surplus is to set

$$\tau_t = -\phi_0 + \phi_1 \frac{G_t}{a_t P_t}, \tag{10}$$

with $\phi_0 > 0$, $\phi_1 > 0$, thus making fiscal policy "passive" if we consider an infinite horizon and no bounds on τ. We consider a deterministic version of this model in which Y remains constant, but there is an upper bound $\bar{\tau}$ on τ_t. We assume that if under the fiscal policy (10) τ_t would exceed $\bar{\tau}$, policy switches permanently to $\tau_t \equiv \bar{\tau}$. At that point, monetary policy changes to $a_t \equiv .07$. In any period where fiscal policy is still passive, monetary policy sets $M_t \equiv \bar{M}$.

Assume $\phi_1 = .07$, $\beta = .95$, $\phi_0 = 1$, $\bar{\tau} = 1$. This makes the steady state level of real debt large enough that the passive fiscal rule implies a τ exceeding $\bar{\tau}$. Thus the economy may start with $\tau_t < \bar{\tau}$, but it cannot stay there indefinitely.

(a) Find conditions on $\bar{M}$ and G_0 such that initial τ_0 is less than $\bar{\tau}$. Show plots of the time paths of P, $G/(aP)$, M and a from these initial conditions.

(b) Show how (if at all) the initial price level varies with ϕ_0 and ϕ_1 in your equilibrium where $\tau_0 < \bar{\tau}$.

(c) Consider how inflation or deflation at the switch date varies with the choice of post-switch constant a_t. Would it be possible for monetary policy to keep prices approximately stable across the date of the switch?

Here's a general approach to the solution. A detailed solution with the calculations and plots asked for will follow later.

After the switch date, which we will call T, the real value of the debt, $b = G/(aP)$, has to remain constant at

$$b = \frac{\beta}{1-\beta}\left(\bar{\tau} + \frac{M}{P}(1 - g^{-1})\right), \tag{11}$$

where g is the gross rate of growth of P and M. That is, real debt is the discounted present value of future real primary surpluses and seignorage.

At $T-1$, real debt b_{T-1} must be close to the value that triggers the shift in policy, which is $b^* = (\bar{\tau} + \phi_0)/\phi_1$, but below it, and it must be true that without the policy switch the debt would exceed the trigger level. The solution depends on what b_{T-1} is relative to b^*. A simple expedient is to suppose that $b_{T-1} = b^*$. Then the period-T government budget constraint gives us

$$b_T = (1 - \gamma f'_{T-1} v_{T-1}^2)^{-1} b_{T-1} \frac{P_{T-1}}{P_T} - \bar{\tau} - \frac{M_T - \bar{M}}{P_T} \tag{12}$$

From FOC's, the definition of v, and the social resource constraint we can get in addition the following equations

$$g = \frac{\beta}{1 - \gamma f'_T v_T} \tag{13}$$

$$1 - \gamma f'_T v_T^2 = \frac{1}{1 + \bar{a}} \tag{14}$$

$$1 - \gamma f'_{T-1} v_{T-1}^2 = \beta \frac{\bar{M}}{M_T} \frac{v_T (1 + \gamma f_T + \gamma f'_T v_T)}{v_{T-1}(1 + \gamma f_{T-1} + \gamma f'_{T-1} v_{T-1})} \tag{15}$$

$$v_T = \frac{P_T C_T}{M_T} \tag{16}$$

$$v_{T-1} = \frac{P_{T-1} C_{T-1}}{\bar{M}} \tag{17}$$

$$C_t = \frac{Y}{1 + \gamma f(v_t)}, \text{ all t.} \tag{18}$$

This list of equations allows us to solve for b_T, v_T, g, P_{T-1}, P_T, M_T, v_{T-1}, C_T, C_{T-1} given values for γ, β, $\bar{\tau}$, $\bar{a}$, $\bar{M}$, and b_{T-1}. Once we have v_{T-1}, can use the equation, derived from the FOC's using $M_t \equiv \bar{M}$

$$z_t = \beta (1 - \gamma f'_t v_t^2)^{-1} z_{t+1}$$

to solve backwards from $T-1$ to any earlier date we wish to consider as a startup date. With $f(v) = v/(1+v)$ solving for v at each date probably has to be done numerically. With $f(v) = v$, it just requires solving a quadratic at each date. Since the real rate of return on government debt is always $1/\beta$ and there is no siegnorage while $M_t \equiv \bar{M}$, it is straightforward to solve the government budget constraint backward in time from $T-1$ to get the starting value of b.

The equation system to be solved for the time $T-1$ values could certainly be made smaller by substitutions, but the resulting algebraic expressions would get longer and more subject to error. This set should be straightforward to solve with a computer equation-solving routine.