Optimal fiscal and monetary policy with distorting taxes

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When government debt pays a lower return than private assets, the reasoning in Friedman's essay on the optimal quantity of money suggests that it would be optimal to expand the debt until its return matched that on private assets. When the only other source of revenue is a distorting tax, however, this is not true. In a perfect foresight model, a benevolent government that can make credible commitments chooses a large gap in returns initially and high distorting taxation in the distant future. The optimal path of taxation is time-inconsistent, with ever-increasing temptation to abandon the path.

Blanchard (2019) and Mehrotra and Sergeyev (2019) have reminded us that real rates of return on government debt have in most years been so low that the debt-to-GDP ratio would decline or remain stable even if the debt were simply rolled over. That is, without any taxation to "back" the debt, with interest and principal payments being financed entirely by the issue of new debt, the debt-to-GDP ratio would not increase. They characterize this situation as one in which there is zero or negative "fiscal cost" to public debt.

The possibility that when debt pays a lower return than other assets permanent deficits are possible has been explored by Reis (2021) and by Mian, Straub, and Sufi (2022), who do not include distorting taxes.

Conventional thinking about public debt sees debt as requiring fiscal backing, so that increased debt requires increased future taxes or reduced future expenditures. If the taxes are distorting, this is a burden. But if we can increase expenditures on beneficial programs, or reduce distorting taxes, without any need to

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offset the resulting deficits with future increased taxes or reduced expenditures, it appears it would be irresponsible not to exploit this possibility.

There is another approach to thinking about optimal fiscal policy that reaches an apparently opposite conclusion, however. Friedman (1969) argued that optimal monetary policy should set the nominal interest rate to zero, thereby making the real return on money match that on other assets. If "money" pays interest, Friedman's prescription is that the interest rate on money should match that on other assets. In many models, this requires taxation, either to pay explicit interest or to contract the supply of outstanding government liabilities. So Friedman's prescription becomes "increase government surpluses to raise the rate of return on government liabilities, until the rate on government liabilities matches that on other assets". Blanchard's reasoning, on the other hand, might be read as "so long as the rate of return on government liabilities is low enough, increased deficits are desirable."

While there are many papers on optimal fiscal and monetary policy, three of them illustrate the range of conclusions and contradictions in this area. Chari and Kehoe (1999a)[1] show that in a model like that in this paper, an optimizing government that makes policy commitments at an initial date, will set the liquidity premium on government debt to zero at all times.[2] Woodford (1990) reaches the apparently contradictory conclusion that in the same sort of model, the optimal steady state may involve a positive liquidity premium. Chari and Kehoe suggest that Woodford's result arises because he considers only steady state equilibria (i.e. with constant tax rates), while their results apply when instead the government chooses a fully optimal, and non-stationary, path for policy. However, in a paper cited by neither the Chari/Kehoe paper nor the Woodford survey paper, Calvo (1978) showed in a very similar model that opti-

[1]The main argument in the Chari/Kehoe paper follows that in the earlier paper by Chari, Christiano, and Kehoe (1996). The earlier paper considered only cash-in-advance models, while the later one covered also money-in-utility-function models, which are equivalent to those in this paper.

[2]Their model, like the main version of the model in this paper, makes setting the liquidity premium to zero impossible, but it can be set arbitrarily close to zero, and welfare improves as the premium shrinks.

mal policy leads to a non-stationary equilibrium in which the liquidity premium does not converge to zero. This result arises despite the fact that the government could by appropriate policy induce a stationary equilibrium with a zero liquidity premium.

Chari and Kehoe argue that their result depends crucially on their assumption that balances and consumption enter the utility function separably, as a homothetic aggregate (so utility is $U(\phi(C,m),L)$ with ϕ homothetic). Calvo considers only models in which utility is additively separable in real government debt and consumption, which does not in general satisfy Chari and Kehoe's homotheticity assumption. It might appear, then, that this is why Calvo's result is different.

As we will see, though, this paper shows that the most important distinction between the Chari/Kehoe paper and the other two is that Chari and Kehoe assume the government has the option at "time zero" of backing the entire money stock with interest-bearing private-sector liabilities. This then allows financing the necessary interest payments on debt, or the contraction of the nominal stock, without any need to use distorting taxes. The papers by Woodford and Calvo, like this paper's model, assume instead that the government cannot hold large amounts of privately issued assets.

This paper works with a model that matches the assumptions on homotheticity and separability in Chari/Kehoe, matches their assumption that saturation of demand for liquidity is possible only asymptotically, and considers a possibly non-stationary Ramsey solution. The difference that alters the result is entirely in this paper's assumption that the government cannot hold private sector bonds that pay a premium return over government debt.

This paper's model reproduces Calvo's result that a benevolent government that can make firm commitments to future policy optimally initially runs deficits and generates a large gap between the return on government debt and the discount rate. The optimal policy promises an ever increasing tax rate. This type of result is probably generic. Postponing distorting taxes makes the required amount of taxation rise, but only at the real rate on government debt, which is less than the discount rate. So unless the distortion costs are rising rapidly with the tax rate, discounted utility rises when tax increases are postponed this way. In models like this one (and Calvo's) that assume perfectly flexible prices, optimal policy is likely to generate immediate inflation, while in a sticky-price model the inflation would be delayed. The same mechanism supporting delayed tax increases would nonetheless probably apply.

Angeletos, Collard, and Dellas (2022) considers a system close to Calvo's, but provides deeper discussion of potential microfoundations for it. The Angeletos *et al* paper also assumes real debt is taken by the government as given at time 0. It therefore does not consider the possibility of a jump in the price level at the initial date and does not get the result in this paper and Calvo's that optimal initial real debt is uniquely determined.

Section I sets out a simple model in which real debt enters the objective function homothetically, but the government can borrow, but not lend. We use this model in section II to discuss what kinds of liquidity benefits specifications might support optimality of a steady state with no liquidity premium and in section III to discuss determination of initial conditions by the optimizing planner. In section IV we give the liquidity technology function an explicit linear form and show how the steady state solutions vary with the levels of government spending and liquidity benefits. Section V shows how the results from this model without growth can be interpreted as results for a model with growth via a simple rescaling. In section VI we consider how the price level and optimal policy choices react to one-time, unanticipated changes in tax rates or liquidity benefit levels. Section VII displays the optimal non-stationary equilibrium path for a particular set of parameter values. Section VIII considers how results would be affected by limited ability of policy-makers to deliver on commitments. The final two sections discuss how broadly applicable the paper's results are. These sections consider parametric restrictions that could be relaxed and omitted real world complications that could be explored.

I. Simple model

A representative individual chooses the time paths of C (consumption), L (labor), A (nominal government debt) and B (possibly interest-bearing government debt) while taking P (the price level) and τ (the tax rate on labor) paths as given (and known in advance). The individual's objective is to maximize

$$\int_0^\infty e^{-\beta t}(\log C_t - L_t)\,dt \tag{1}$$

subject to

$$C \cdot \big(1 + f(v)\big) + \frac{\dot{B} + \dot{A}}{P} = (1-\tau)L - \phi + \frac{iB}{P} + \frac{rA}{P} \tag{2}$$

$$v = \frac{PC}{B}, \quad B \geq 0, \quad A \geq 0, \quad L \geq 0\,. \tag{3}$$

Here v is a version of "velocity", and $f(v)/(1 + f(v))$ can be interpreted as the fraction of consumption spending that is absorbed by transactions or liquidity costs. (Henceforth we will treat "transactions costs", "liquidity costs" as equivalent and "liquidity benefits" as minus the same thing.) B is government liabilities that provide a transactions or liquidity service, while A is government liabilities valued only for their return. Both may pay interest, but unless demand for liquidity or transaction services is saturated, $i < r$. τ is the proportional tax rate on labor, and ϕ is a lump-sum tax.

An equivalent specification would define

$$C^* = C \cdot (1 + f(v))$$

and then make utility depend on C^* and $b = B/P$, while omitting the $f(v)$

term in the budget constraint. Note that if utility is of the form $U(C, L)$, and we rewrite the problem with C^* but no v term in the budget constraint, and $V(C^*, b, L) = U(C, L)$ in the objective function, we have, by making B/P enter the definition of C^* only via $f(v)$, enforced homotheticity of V in $C^*, B/P$. This is important for connecting this paper's results to Chari and Kehoe (1999b), who emphasize that their results depend on homotheticity of utility in C^* and b.[3] The money-in-budget-constraint formulation makes it easier to see what fraction of spending is absorbed by transactions costs, and in particular avoids allowing implied transactions costs becoming negative.

The linear disutility of labor and unbounded supply of labor are chosen to make the algebra of deriving our results simpler. As should become clear, the results don't depend on these details of the specification.

The private agent's problem leads to first order conditions

$$C \cdot (1 + f + f'v) = 1 - \tau \tag{4}$$

$$\frac{-\dot{\tau}}{1-\tau} = i + f'v^2 - \beta - \frac{\dot{P}}{P}\,. \tag{5}$$

$$r - i = f'v^2\,. \tag{6}$$

We assume $f'v^2$ is monotone increasing in v. This is automatically true if $f''(v) \geq 0$ for all $v > 0$, but if we want to allow for the possibility of equilibria with $b = 0$, $f(v)$ must remain bounded as $v \to \infty$. This is possible with $f'v^2$ monotone increasing, but there are reasonable-looking choices of f that violate monotonicity of $f'v^2$. We rule them out because they raise the possibility of multiple equilibria, which would complicate analysis of the model.

The government has to finance a given path G of expenditures, which provide

[3]For example, with $f(v) = \gamma v$ and this model's $U(C, L) = \log(C) - L$, we get

$$V(C^*, b, L) = \log\left(\left(\sqrt{1 + \frac{4\gamma C^*}{b}} - 1\right) \cdot \frac{b}{2\gamma}\right) - L\,.$$

no utility to the private agents. Its budget constraint is

$$\frac{\dot{B}+\dot{A}}{P}+\tau L+\phi=G+\frac{iB+rA}{P}\,. \tag{7}$$

The social resource constraint

$$C\cdot(1+f)+G=L \tag{8}$$

is derivable from the government and private budget constraints.

II. Conditions for $r = i$ in optimal steady state

Saturating demand for liquidity, as suggested by the Friedman rule, is possible if $f(v)$ is zero over some interval $(0, v^*)$ and then starts increasing. We can see that there are conditions on f that imply that the optimal steady-state equilibrium does not saturate liquidity demand, and also conditions under which saturating liquidity demand is at least a local optimum among steady states. Assume that $A = 0$, i.e. all government debt provides liquidity services. (We will show below that an optimizing government will always choose policies that make $A = 0$.) Then the government budget constraint (7) implies that in steady state

$$b=\frac{\tau L+\phi-G}{\rho}\,, \tag{9}$$

where $\rho = i - \dot{P}/P$ is the real return on government debt. From (5) we have that in steady state

$$\rho=\beta-f'v^2\,. \tag{10}$$

Proposition. *If in a steady state equilibrium with constant $\tau > 0$ and ϕ*

a) $f(v) = 0$ *for* $0 \le v \le v^*$,

b) $f(v)$ *and* $f'(v)$ *are both positive for* $v > v^*$, *and*

c) $f(v^*) = f'(v^*) = f''(v^*) = 0$,

the demand for liquidity is not saturated (i.e. $v > v^*$*) in the optimal steady state.*

PROOF:

Equations (4)-(6), (8), (9) and the definition $v = C/b$, when they have a solution for C, v, b, L, ρ and L, define a steady-state equilibrium determined by the constant fiscal parameters τ an ϕ. From (4) we can see that, with the assumed conditions on f and its derivatives at v^*,

$$\frac{dC}{d\tau} = \frac{dL}{d\tau} = -1 \quad \text{at} \quad v = v^* .$$

The derivative of steady-state utility $\log C - L$ with respect to τ is then

$$\frac{-1}{1-\tau} + 1 < 0 ,$$

which implies that steady state utility can be increased by reducing the labor tax rate.

Reducing taxes raises consumption and labor input, with these effects netting out to increase utility. Reducing taxes also reduces the equilibrium level of real debt b, which reduces $f(v)$. The assumption that $f''(v^*) = 0$ means that this latter effect is negligible for small reductions in taxes.

It is certainly possible to choose a form for f and parameter values so that the optimal steady state equilibrium does set $\beta = \rho$. If the second derivative from the right of f, $f''(v^*)$ is large enough, transactions costs rise so rapidly as v increases that the positive effect on transactions costs of reduced τ dominate, and $v = v^*$ becomes optimal.

III. What are reasonable assumptions about initial conditions?

The government budget constraint (7) implies that total government liabilities, $A+B$ has a continuous time path. In other words, to change $A+B$ requires changing tax revenues $\tau L+\phi$, government expenditures G, or interest payments on debt $iB+rA$. It is unreasonable to think of any of these as being changeable at infinite rates. Treating the rate of growth $\dot{A}+\dot{B}$ as well defined at every date, which is implied by treating (7) as a non-forward-looking equation, makes economic sense.

The government budget constraint does not require that A and B individually have continuous time paths. With B_{-0} and B_{+0} (for example) as notation for the left and right limits of B at time zero, we have

$$B_{-0}+A_{-0}=B_{+0}+A_{+0}\,, \tag{11}$$

while $B_{-0}\neq B_{+0}$ and $A_{-0}\neq A_{-0}$ are possible. Private agents see themselves as having to save out of current income to change their total assets $A+B$, but see themselves as able to trade instantly with the government to any desired ratio of A to B in their asset holdings. But then if $B_{-0}+A_{-0}$ is non-zero, to arrive at zero net worth at time zero requires a policy that makes P_{+0} infinite. If A is bounded below, this implies in turn real balances $b_{+0}=B_{+0}/P_{+0}$ are zero and that transactions costs are maximal.

While setting interest bearing debt to zero (or its minimum feasible value) is beneficial, in the model money balances are important. Unless barter equilibrium ($b=0$) is optimal, an optimizing government, free to choose a time path of policy that fixes the price level at time zero, will choose a combination of inflation and tax policies that make $B_{+0}=B_{-0}+A_{-0}$ and thus $A_{+0}=0$.

Another way to explain our initial date assumption is that we assume the government issuing nominal debt is recognized to have the option at any date of diluting the claims of holders of the debt and of money balances by deficit finance, but is assumed to be committed not to introduce a new currency, and debt denominated in it, that has a higher seniority as a claim on future revenues.

Of course if the government at time zero can repudiate both existing debt and existing currency, then issue new currency and use it to buy private sector liabilities, and if it can do all this while preserving its perfect credibility, it can achieve a better outcome than is obtainable by just setting the P_{+0} so that agents choose $B_{+0} = 0$. However defaulting on outstanding debt and currency is an action that the government is assumed never to do in the future. On the other hand using unanticipated price level fluctuations to stabilize the real value $b + a$, and maintaining $a \doteq 0$, is not only optimal at time zero, in a stochastic version of the model the fully committed government does this at every date.

In mapping the conclusions of this model into an actual economy, the optimality of $a \equiv 0$ should not be taken as implying that interest-bearing debt does not or should not exist. In reality government debt comes in many maturities, all or most of which pay real returns lower than what is apparently available on other investments. The $b = 0$ conclusion means that government debt that exists should pay an interest rate less than or equal to what is available on other investments, with equality only when demand for liquidity has been saturated.

If we removed the constraint that $A \geq 0$, we could implement Chari and Kehoe's conclusion on optimal government actions at the initial date: repudiate (or inflate away entirely) the existing $A_{-0} + B_{-0}$, then issue new liquidity-providing debt by purchasing private sector liabilities, so that $A_{+0} + B_{+0} = 0$. Then a policy that sets $\rho = \beta$ can be financed without any use of taxation. But note that the initial step here, the repudiation of outstanding debt at time 0, is one that an optimizing government will be tempted to repeat, but must promise never to repeat.

IV. Numerical examples of optimal steady states

In this section we solve numerically for optimal stationary equilibrium with a given constant government spending G and lump-sum tax ϕ. If ϕ were freely variable, it would be optimal to set $\tau \doteq 0$ and nearly saturate the demand for liquidity. We include ϕ in the model because it may clarify thinking about conditions for existence and uniqueness of equilibrium, which we take up in the appendix. But for numerical solutions we set $\phi = 0$ for convenience.

Krishnamurthy and Vissing-Jorgensen (2012) present evidence that US Treasury securities have historically paid a lower yield than comparable corporate bonds, that this cannot be explained by default risk, and that the yield spread shrinks as the supply of treasury securities increases. They argue that Treasuries behave much like non-interest-bearing high-powered money in these respects. Our model in this section builds on their observations, treating all debt as providing liquidity services. Of course we are simplifying by having only one kind of debt, with a single interest rate and maturity. The argument of the previous section, supports the idea that an optimizing government would issue only liquidity-providing debt.

To allow explicit numerical solution, we specify a parametric form for f, $f(v) = \gamma v$

The optimizing government sets constant values for the labor tax rate τ and i, the nominal interest rate on B. It maximizes the equilibrium value of the private agent's objective function (1), which in this case, restricted to steady states, is the same as maximizing the undiscounted $U(C, L) = \log(C) - L$. The government takes (4), (5), (7) and (8) as constraints. Equation (5) with our linear form for f becomes

$$i = \beta + \frac{\dot{P}}{P} - \gamma v^2 - \frac{\dot{\tau}}{1-\tau}\,. \tag{12}$$

We define

$$\rho = i - \frac{\dot{P}}{P} \,, \tag{13}$$

which is the real rate of return on government debt.

We can rewrite the system with ρ replacing i, and, because i and $\dot{P}/P$ enter the system only as their difference, we then have a system of 5 equations in 5 unknowns, which we assemble here for convenience:

$$C \cdot (1 + 2\gamma v) = 1 - \tau \tag{14}$$

$$\dot{b} = \rho b + G - \tau L \,. \tag{7}$$

$$C(1 + \gamma v) + G = L \tag{15}$$

$$\rho = \beta - \gamma v^2 - \frac{\dot{\tau}}{1 - \tau} \tag{16}$$

$$v = \frac{c}{b} \,. \tag{17}$$

Note that by using ρ, we have eliminated both i and $\dot{P}/P$ from the system. This implies that given the time path of τ, i affects the equilibrium only by changing the inflation rate. Changing i has no effect on the other five variables: C, v, ρ, L, and b. Also, though $\dot{\tau}$ appears in the system, it of course drops out when policy fixes a constant τ, so that equilibrium is defined by 5 ordinary equations in 5 unknowns.

As we will see below, if the optimizing government discounts future utility at the same rate as the representative agent, keeping τ constant is not optimal. We are in this section finding equilibrium with constant τ, and optimizing the constant τ, which is optimal only for a government that does not discount future utility, even though the representative agent does so. An argument in the appendix shows that transversality and feasibility guarantee that when τ is constant, b must also be constant.

Because b can (via a jump in initial price level P_0) jump at the initial date, there is no transition path from equilibrium with one level of constant τ to another if a one-time, permanent, unanticipated change in τ occurs. In this flex-price model, the economy goes immediately to the new equilibrium. The same is true if there is a one-time, unanticipated shift in a parameter, like the transactions cost or liquidity service parameter γ, so long as τ is constant.

The equation system for the constant-τ case can be solved analytically, resulting in a univariate quadratic equation. With $\gamma = \tau = G = 0$, the agent chooses $L = C = 1$. Existence of constant-τ equilibrium requires $\tau > 2G - 1$. In that case, there is a unique steady state for every feasible τ, and there are no non-steady-state equilibrium paths[4], so the initial price level is uniquely determined. Note that $\tau < 1$, because otherwise it is optimal to set $L = 0$, and that therefore constant-τ steady states require $G < 1$.

In the tables below τ is the optimal constant τ for the table's assumed value of G. It was found by a one-dimensional grid search on τ. In all the tables $\beta = i = .02$, which means that the gap between β and ρ is generated entirely by inflation.

γ	C	b	L	$\dot{P}/P$	U	τ	$psurp$	$\frac{\gamma v}{1+\gamma v}$	P_0
0.001	0.972	1.31	0.973	0.0006	-1.001	0.026	0.0007	0.0007	0.76
0.010	0.940	2.70	0.943	0.0012	-1.005	0.054	0.0033	0.0035	0.37
0.100	0.868	5.27	0.882	0.0027	-1.024	0.103	0.0143	0.0162	0.19
1.000	0.721	9.27	0.778	0.0061	-1.104	0.166	0.0561	0.0722	0.11

TABLE 1—OPTIMAL STEADY STATE WITH $G = 0$, $\beta = .02$, $i = .02$

γ: transactions cost parameter; C: consumption; b: debt; L: labor; $\dot{P}/P$: inflation rate; U: utility; τ: labor tax rate; $psurp$: primary surplus; $\gamma v/(1 + \gamma v)$: proportion of consumption expenditure absorbed by transaction costs; P_0 : initial price level, assuming $B_0 = 1$; G non-productive government expenditure; β: discount rate.

Tables 1 2, and 3 show the model's optimal steady state at various levels of

[4]As is shown in appendix A, uniqueness may depend on trigger policies, or small positive values for ϕ that are negligible in equilibria but important on off-equilibrium paths.

γ	C	b	L	$\dot{P}/P$	U	τ	$psurp$	$\frac{\gamma v}{1+\gamma v}$	P_0
0.001	0.74	0.28	0.99	0.0069	-1.29	0.26	-0.0037	0.0026	3.56
0.001	0.972	1.31	0.973	0.0006	-1.001	0.026	0.0254	0.0007	0.76
0.010	0.940	2.70	0.943	0.0012	-1.005	0.054	0.0507	0.0035	0.37
0.100	0.868	5.27	0.882	0.0027	-1.024	0.103	0.0912	0.0162	0.19
1.000	0.721	9.27	0.778	0.0061	-1.104	0.166	0.1293	0.0722	0.11

TABLE 2—OPTIMAL STEADY STATE WITH $G = .25$, $\beta = .02$. $i = .02$

See notes to Table 1

γ	C	b	L	$\dot{P}/P$	U	τ	$psurp$	$\frac{\gamma v}{1+\gamma v}$	P_0
0.001	0.20	0.022	0.998	0.0823	-2.627	0.800	-0.0013	0.0090	46.27
0.010	0.19	0.064	0.994	0.0855	-2.665	0.801	-0.0042	0.0284	15.55
0.100	0.17	0.171	0.983	0.0955	-2.775	0.801	-0.0129	0.0890	5.87
1.000	0.12	0.365	0.965	0.1137	-3.060	0.794	-0.0342	0.2522	2.74

TABLE 3—OPTIMAL STEADY STATE WITH $G = .8$, $\beta = .02$, $i = .02$

See notes to Table 1

government spending G and transactions cost parameter γ. Even in Table 1, where there is no government spending to finance, it is optimal to have a positive rate of labor tax τ, in order to induce low inflation, a positive real return ρ on government debt, and hence a reduced incentive to conserve on real balances. However even in this case the inflation rate does not go to zero, as would be required to make $\rho = .02$ and thereby implement the Friedman rule.

The column labeled $purp$ is the steady state primary surplus, $\tau L - G$. Steady states with negative primary surplus exist in this model, even though in its simple form it has no growth. A permanent primary deficit arises when the real rate of return on government debt is negative, and by choosing a low enough constant τ, the return can be driven negative. However as can be seen from the three tables, the primary surplus turns negative here only when G is very large, in Table 3.

The most realistic of these cases, might be Table 2's $\gamma = .01$ row. This produces a "debt to GDP" ratio of about 0.8, transactions costs absorbing about 1% of consumption and a tax rate of 27%. In this case the optimal real rate on debt ρ (the 2% nominal rate i minus the inflation rate) is 1.24%.

In Table 3 taxes and inflation substantially depress consumption in order to accommodate the high level of G. Inflation is over 8%. Nonetheless the steady state primary deficit delivers only a small fraction (about half of one percent) of total revenue.

With $\tau = 0$, the primary deficit is increasing in v over the whole $(0, \infty)$ range, and approaches .5 from below as $v \to \infty$. Even in Table 3, where government expenditure is taking up around 80% of output and the tax rate is about 80%, the primary deficit remains well below its upper bound. Even though the Friedman rule is not optimal in this model, the inflation tax produces little revenue relative to the distortion it induces, so that it is optimal to use it to generate revenue only when the other tax available is at highly distortionary levels.

V. Allowing for growth

The arithmetic of "zero fiscal cost" debt might seem to rest on there being positive growth. The idea is that if the economy is growing at a rate exceeding the real rate of return on government debt, debt can be increased today, without any actual or expected future increases in taxation (or reductions in expenditure). with debt nonetheless shrinking relative to output over time.

This line of reasoning is misleading. It ignores wealth effects and inflation. People must be induced to hold the additional debt. They can be induced to do so, even without increased taxation, but only by lowering its real value through inflation. This is true whether or not there is economic growth.

In fact, in this paper's model if we introduce constant labor augmenting technical progress at the rate θ and interpret C, G and $psurp$ as ratios to $e^{\theta t}$, the model is unchanged except that the discount rate now has to be interpreted as the discount rate β of private agents plus the rate of technical progress θ. None of the table entries then change. The "β = .02" line in the captions would change to $\beta + \theta = .02$.

Of course, tables calculated with private agents' $\beta = .02$, $\theta = .02$, would differ from those shown. But the difference would not be lower labor taxes at a given G. Because the Friedman rule requires a higher return on B with an increase from β to $\beta + \theta$ in the discount rate, the optimal steady state is likely to have a positive primary surplus (that is, $\tau L > G$) even with positive θ. If we start with the economy of the second row of Table 2 and increase θ from 0 to .02 or .04, the optimal steady state still implies a positive primary surplus, and the primary surplus is larger for larger θ. The increased liquidity costs from a greater gap between ρ and $\beta + \theta$ more than offset the benefits from the reduction in τ that would be possible with a permanent primary deficit.

VI. Effects of a jump in liquidity demand or in the tax rate

The model, as we have noted, goes to its steady state immediately when G and τ are fixed. We can see the consequences of going from the optimal steady state with $G = .2$ to the optimal steady state with $G = .25$ by comparing lines 1 and 2 of Table 4.

G	τ	C	b	L	$\dot{P}/P$	U	$psurp$	$\frac{\gamma v}{1+\gamma v}$	P_0
0.20	0.220	0.768	1.00	0.974	0.0059	-1.238	0.0142	0.0076	1.00
0.25	0.266	0.721	0.83	0.977	0.0076	-1.304	0.0103	0.0086	1.21
0.20	0.266	0.729	2.51	0.932	0.0008	-1.247	0.0481	0.0029	0.40
0.25	0.220	0.712	0.15	0.996	0.2281	-1.336	-0.0310	0.0456	6.71

TABLE 4—OPTIMAL AND SUBOPTIMAL FINANCING OF G

See notes to Table 1 for variable definitions. $\gamma = .01$ for all lines. Lines 1 and 2 show solutions with optimal tax rates for the given G values. Lines 3 and 4 are solutions for given G and τ, with no optimization. Comparing lines 1 and 4 shows the change in going from $G = .20$ with optimal τ to $G = .25$ with unchanged τ. Comparing lines 2 and 3 shows the reverse case.

The table shows that it is optimal to have a positive primary surplus in both cases, with the change in G almost entirely covered by a change in τ. The effects on C, L and utility U are minor. However the effects on b and P_0 are substantial. Optimal policy requires reducing b by nearly 20 per cent, and accomplishing this mainly through a jump in the price level of over 20 per cent. Within the model, this jump in the price level is costless, but in a richer model with sticky prices it would be costly. That it would be costly does not necessarily imply that in a model recognizing price stickiness a rapid temporary inflation would not be part of an optimal solution. Avoiding the jump would require credibly announcing a decreasing time path for the tax rate.

Comparing lines 1 and 4 shows that increasing G without any adjustment in τ would greatly increase both the initial jump in prices and the resulting steady state inflation rate. Thus it is possible to increase G and finance the increase entirely without change in tax rates, but only at the cost of high inflation. Note that it is only in line 4, where expenditure is sub-optimally increased without any tax increase, that the steady state shows a primary deficit.

We can do a similar exercise comparing lines 1 and 2 or lines 2 and 3 of Table 2. This lets us see the optimal policy response to a sudden increase in the demand for B. Going either from line 1 to 2 or from line 2 to 3, we see that the values of steady state C, L τ and U are little changed by the rise in demand for government liquid assets. But P_0 and b are drastically affected. The optimal response to the rise in liquidity produces a downward jump in P_0 by a factor of roughly 3. Unlike an upward jump in P_0, it seems possible that the jump could be eliminated by a feasible fiscal intervention. What would be required is a sudden expansion of B_0 at the initial date, in proportion to the drop in P_0 that would be required without any change in B_0. This could be accomplished by a "helicopter drop" — lump sum transfers of government paper to the public.

It is tempting to take this case of response to a rise in demand for liquidity as roughly corresponding to the policy problems of the US (and other countries) in the wake of the Great Financial Crisis. In this interpretation, the rise in outstanding real debt was required to avoid rapid deflation. With debt expansion arising from this source, the lack of any rise in inflation or interest rates is not surprising.

The expansion is not in itself a reason for greatly increased fiscal stringency. Just as before the increase in real debt, the desirability of unbacked debt finance, i.e. of increased reliance on the $\beta - \rho$ gap, depends on comparing the distortion from constricting the supply of liquidity services from government debt to the reduced distortion from direct taxation, or the benefits from productive government expenditures, that would be allowed by increasing reliance on low returns on government debt. So long as inflation, interest rates on government debt, and tax rates remain as they were before the debt expansion, the optimal balance of seignorage vs. direct taxation is not much affected by the level of debt.

On the other hand, on this interpretation, the fact that interest rates and inflation have been stable in the face of the recent rise in debt cannot be taken as implying that debt financed fiscal expansion can be continued without limit without inflationary consequences.

VII. Optimal, time-inconsistent policy with full commitment

The discussion of steady state equilibria in the previous few sections may suggest that optimal policy makes little use of the option to run permanent primary deficits and thus pushes the real rate on government debt close to the discount rate. But if the government discounts the future at the same rate as private agents, a benevolent government that can commit to a time path for policy will not find a steady state equilibrium to be optimal. It will in fact optimally run primary deficits initially, while promising higher taxation in the future. The deviations of this optimal non-stationary policy from the optimal steady state policy are initially large.

The equation system on page 12 can be solved to allow expressing all variables in the system as functions of v, τ, and $\dot{\tau}$. Dividing the government budget constraint by b and expressing the left-hand side ($\dot{b}/b$) and ρ as functions of τ, $\dot{\tau}$ and

v produces

$$\frac{\dot{b}}{b} = \frac{-\dot{\tau}}{1-\tau} - \frac{\dot{v}}{v} \cdot \frac{1+4\gamma v}{1+2\gamma v} = \beta - \gamma v^2 - \frac{\dot{\tau}}{1-\tau} + \frac{G - \tau L}{b} . \tag{18}$$

The terms in $\dot{\tau}$ cancel, allowing us to derive an expression for $\dot{v}/v$ in terms of τ and v alone:

$$\frac{\dot{v}}{v} = \frac{\overbrace{1+2\gamma v}^{S}}{1+4\gamma v} \overbrace{\left(\gamma v^2(1+\tau-2G) + (\tau - G)v - \beta\right)}^{R} . \tag{19}$$

The two expressions in the formula for $\dot{v}/v$ are labeled R and S to aid understanding of the code for the government's first order conditions. These two expressions and their derivatives play a central role in that code. Note that the locus of constant-v points is defined by $R = 0$.

Equation (19) captures all the constraints on the government imposed by the social resource constraint and private agent choice behavior. An optimizing government with the same objective function as the private agent will therefore solve this problem:

$$\max_{\tau, v} \int_0^\infty e^{-\beta t} (\log C_t - L_t)\, dt \tag{20}$$

$$\text{subject to (19), (14), and (15)} . \tag{21}$$

The first-order conditions for this problem lead, after using (4) and (8) to leave just v and τ in the system, to two equations determining the optimal path. One involves no derivatives (because $\dot{\tau}$ does not appear in the constraints or the objective function):

$$\frac{1}{1-\tau} = \frac{1+\gamma v}{1+2\gamma v} + (\gamma v^2 + v)\eta \frac{1+2\gamma v}{1+4\gamma v} , \tag{22}$$

where η is the multiplier, or costate, associated with (19).

The other is a messy expression determining $\dot{\eta}$ as a function of η, τ, and v. Computer code that builds up the expression from components is available in the R function `odevtau()`[5] online. The algebraic expression for it is omitted here, as the code is probably easier to follow and verify.

Equation (22) allows us to substitute τ out of both the $\dot{\eta}$ expression and the $\dot{v}$ expression (19), giving us a system of two ordinary differential equations in two unknowns, v and η.

I have not been able to compute analytical solutions to the system, but the system is relatively simple to understand via numerical and graphical methods. One component of the analysis sets up a 100 by 100 point grid in (τ, v) space and calculates $\dot{\tau}, \dot{v}$ pairs at every point on that grid. The locus of (approximate) $\dot{v} = 0$ points on the grid is then the set of constant-v points.[6] This locus is entirely determined by (19) and, since it is not affected by $\dot{\tau}$, is also the locus of constant-τ steady-state solutions displayed in the tables above. The locus of $\dot{\tau} = 0$ points can also be plotted on the same panel, and any intersection of the two lines is a steady state of the planner's problem.

In Figure 1 the green line is the locus of constant-τ steady states, the optimal steady state from the $\gamma = .01$ line of Table 2 is the blue triangle, and the black line, entirely above the green one, is the locus of points where the optimizing planner sets $\dot{\tau}$ to zero. From the figure, it might seem that the two curves do not intersect at all, but this is an artifact of the finite resolution of the grid. Figure 4 zooms in on a thin sliver at the left edge of Figure 1 plot, still using a 100 by 100 grid, and it is clear that the two curves do intersect — at $\tau = 1$. (This can in fact be shown analytically.)

In Figure 1 the red arrows indicate the direction and magnitude of change in (v, τ) at each point, along a path satisfying the local first-order conditions

[5]Available at http://sims.princeton.edu/yftp/CostlessDebt/odevtau.R. This is the code actually used in the final stages of the paper, and works with v and τ as the states rather than v and η.

[6]This locus easily plotted using the R **`contour()`** function.

(Euler equations) of the planner's problem. The planner can choose the starting point of the economy's path. By announcing the time paths of τ and deficits, the planner determines the initial price level and, thereby, both v and τ at the initial date. With the initial state freely chosen, the optimum is determined by the initial transversality condition $\eta_0 = 0$, combined with restrictions on the path's behavior as $t \to \infty$.

The red arrows on Figure 1 make it clear that when starting from points above the $\dot{v} = 0$ line, these Euler equation paths imply rapid increase in v, with τ converging to the $\dot{\tau} = 0$ line. In fact it can be shown that along such paths $v \to \infty$ in finite time. Since $v \to \infty$ implies $C \to 0$, paths like this have infinitely negative discounted utility and thus cannot be solutions to the optimizing government's problem — we know there are feasible constant-τ paths that deliver finite discounted utility.[7] Paths that start well below the $\dot{v} = 0$ line lead to steadily declining v, toward $v = 0$. These paths are ruled out by the private agents' transversality condition: as $v \to 0$, an optimizing agent can eventually improve on such a path by consuming part of her stock of b. This raises future transactions costs, but only slightly if v is close to zero, while consuming a given fraction of b produces arbitrarily high utility as $b \to \infty$.

There is only one path (the "saddle path") that satsifies the Euler equations while neither diverging toward $v = 0$ nor diverging to $v = \infty$. The start of such a path is shown as the orange line on Figure 1. It was calculated by imposing $\eta_0 = 0$ and varying v_0[8] to find a line that did not diverge either to the left or the right.[9] If we could calculate the full path, it would converge to $\tau = 1$, but beyond the part of it shown in orange, it would on the Figure 1 graph seem to

[7]There is a separate question as to whether simply announcing a time path for current and future τ is enough to guarantee a unique initial price level. We take this up in Appendix C. Ruling out explosive paths for the price level usually requires that policy commit to higher taxes if the initial price level is above that consistent with the saddle path.

[8]This method is called "multiple shooting".

[9]Actually, *every* path calculated diverged either to the left or the right. What is shown is just the longest path the differential equation solver could calculate before it diverged. Changes in the initial conditions in the fourth significant digit made the solution change from the path shown (which actually crosses the green line and shoots off to the right) to a path that parallels the green line for a while then shoots off to the left.

coincide with the green line. They are so close, because in this range τ on the optimal path is increasing *extremely* slowly. The part of the optimal saddle path shown as the orange line is traversed over approximately 400 years. Though the economy proceeds, if the time-0 policy commitments are honored, toward $\tau = 1$, it does not get close to $\tau = 1$ for many centuries.

The time paths of several of the model's variables are shown in Figure 2, for the first 400 years, and in Figure 3 for the first 10 years. Inflation is high at the start, taxes are very low, and utility is higher than in the steady-state optimum over the entire first 10 years.

In Calvo's model with real balances entering separably in the utility function, it is optimal to set taxes to zero at time 0. In this model, with $\beta = .02$, $\gamma = .01$, it turns out that optimal $\tau_0 < 0$, i.e. a labor subsidy. The optimal path delivers considerably higher utility in the initial, low-tax periods, and utility approaching $-\infty$ in the far distant future. τ rises above the optimal undiscounted optimal steady state value, but not for a long time, and the rise of τ toward 1 is very slow. The near-term benefits of low taxes more than offset the heavy, but long-delayed, costs of high taxes in the distant future.

To visualize the last part of the optimal path, Figure 4 zooms in on the sliver of Figure 1 at the left edge, for v in (0,.1). There is no separate orange line, because it so nearly coincides with the green $\dot{\tau} = 0$ line. Progress toward $\tau = 1$ in this region is extremely slow, and $\eta \to \infty$ in this region. That is, the benefit of violating the time-zero commitment to this path becomes very large. It is likely that the temptation for a policy maker to re-initialize by dropping taxes would be overwhelming. Note that this is different from the simple Phillips curve examples of time-inconsistency, where the benefits of abandoning commitment to no surprise inflation are constant over time after the first period.

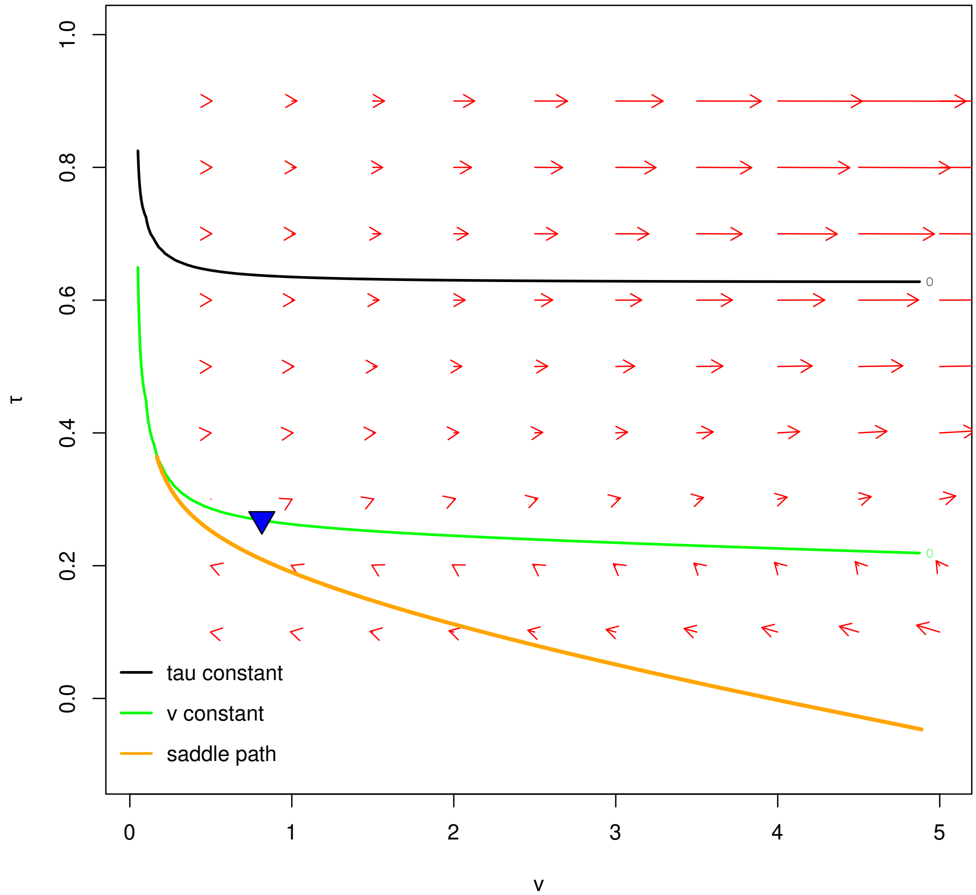

FIGURE 1. BEGINNING OF OPTIMAL PATH WITH .$\beta = .02$, $\gamma = .01$, AND $G = .25$

Note: While the orange saddle path appears to hit the green constant-v locus, in fact it stays extremely close to it, but always below it, and the orange, green and black lines all meet at $\tau = 1$, never crossing before that point. The red arrows show the derivatives of the v, τ vectors. The derivative vectors change rapidly in the neighborhood of the green $\dot{v} = 0$ line. In fact they parallel the green line at points just below it. The blue triangle is the optimal steady state, corresponding to the $\gamma = .01$ line of table 2.

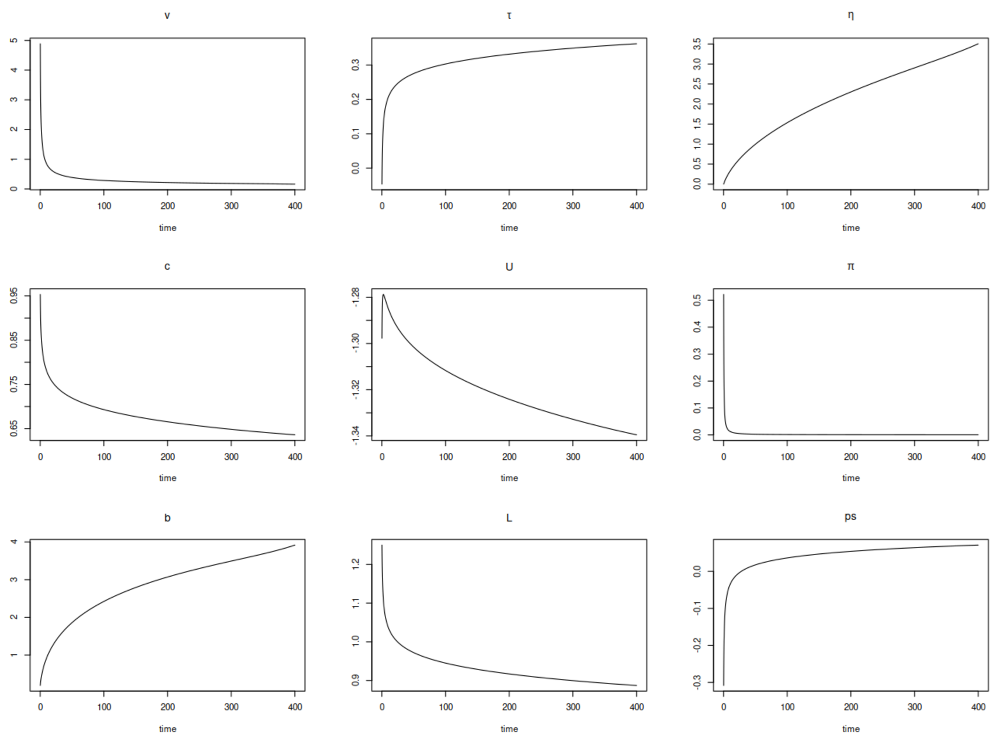

FIGURE 2. 400 YEARS ALONG THE SADDLE PATH

Note: v: velocity (C/b); τ: tax rate; η: costate; C: consumption; U: utility; π: inflation; b: real debt; L: labor input; ps: primary surplus. In the optimal steady state, v=.85, C=.72, U=-1.30, pi=.0072, b=.81.

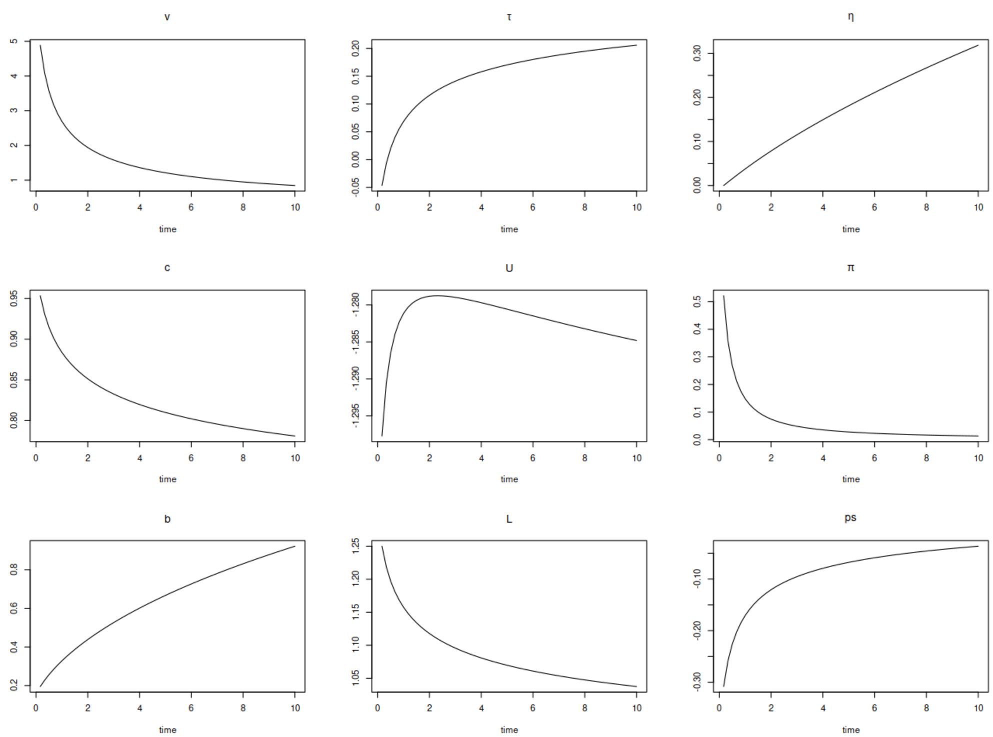

FIGURE 3. 10 YEARS ALONG THE SADDLE PATH

Note: See note to figure 2

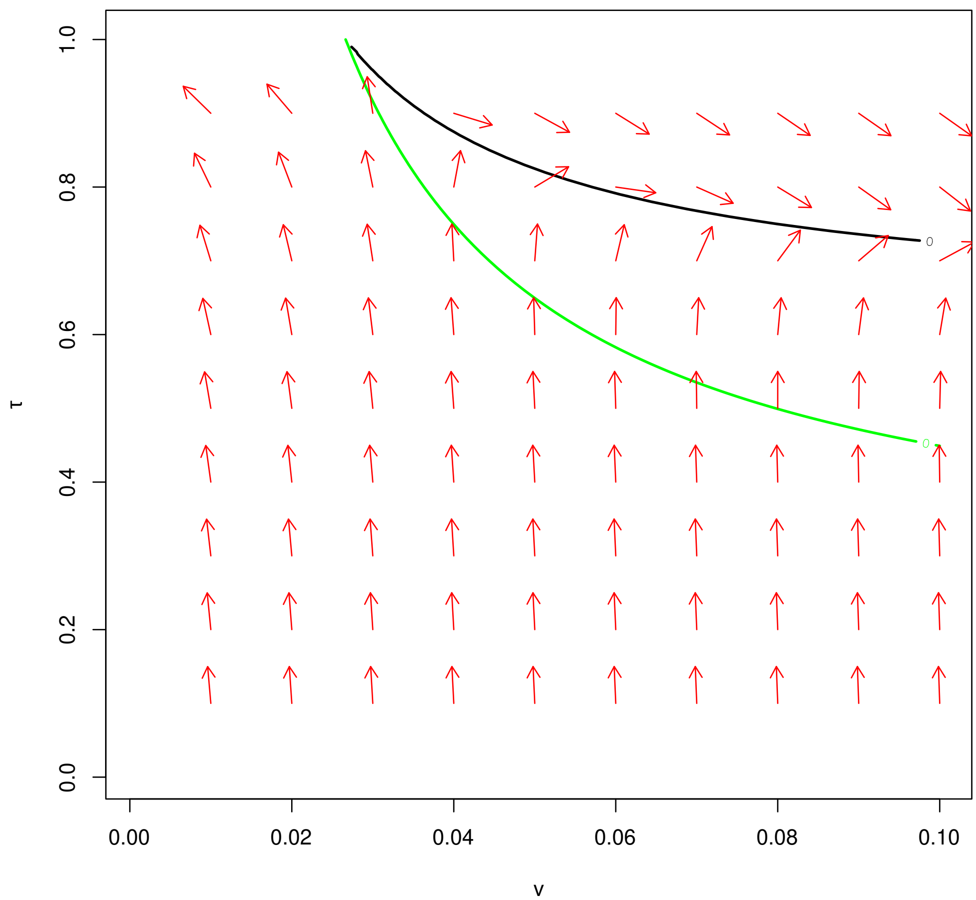

FIGURE 4. END OF OPTIMAL PATH

Note: The arrows on this plot show the directions of the derivative vectors, but not magnitudes. As in Figure 1, the green line is the $\dot{v} = 0$ locus and the black line the $\dot{\tau} = 0$ locus. The optimal path almost coincides with the green line, lying barely below it.

VIII. Limited commitment

The infinite horizon outcome with full commitment is unlikely to be even approximately implemented in practice because of the extremely strong and growing incentive to deviate from it as time goes on. On the other hand assuming no ability at all, even for an instant, to commit to a policy seems equally unrealistic. Realistic models with limited ability to commit would be much more complicated than this paper's setup, but we can gain some insight from this paper's model if we assume an ability to commit for a fixed, known span of time.

Suppose the policy maker remains in office for T years, after which a new policy maker, unbound by the previous policy regime's commitments, takes over, again for a new span of T years, with this rotation of policy makers going on forever. In most models of this type calculating optimal behavior depends on determining how the state variables at the end of a regime affect behavior of the policy maker in the following regime, but in our setup, because the new regime can start with a jump in the price level, there is no effect of the terminal state at time T on the choices of the new policy maker. Each policy maker has an initial transversality condition that makes $\eta = 0$.

However, there can be no jump in the price level with the regime change. The regime changes are deterministic and known by the public. A jump in the price level after the initial date would imply an infinite positive or negative rate of return on government debt at the regime-switch date. Private agents would attempt to buy or sell debt before the price-jump date and undermine the equilibrium. Note that this constancy of the price level across the regime switch date is not a constraint on the new policy-maker's behavior. The new policy maker is free to cause a jump in the price level. The constraint is on the previous regime: the price level at the end of the previous regime must be exactly the value that the new regime's policy will choose, so there is no jump. Since the level of nominal debt B is also assumed not to be able to jump, we can characterize this constraint equivalently as requiring that real debt b not jump. Then, since all policy makers have the same objective function and constraints, the equilibrium chosen by the policy maker in each regime is the same, and b in equilibrium takes the same value at all policy regime-switch dates.

We can calculate the equilibrium in this limited-commitment model using the same set of differential equations we used in the infinite-horizon, full commitment model, replacing the non-explosiveness terminal condition with a constraint on the terminal value b_T. Among optimal paths over $(0, T)$ satisfying $\eta_0 = 0$ and $b_T = b^*$, we choose the one with $b_0 = b^*$, reflecting our knowledge that all policy makers are identical.[10]

Figure 5 shows time paths of model variables when $T = 10$. Compared with the first 10 years of the full commitment solution, the initial inflation is even higher. The tax rate rises much more rapidly. Though utility $U(C, L)$ briefly goes above the steady-state optimal level, the average over the 10 years is much lower than the optimal steady state value, whereas the full-commitment solution delivers higher utility than the steady-state solution over the first 10 years.

That the outcome is worse when commitment is possible only over a finite span is not surprising. Perhaps not so obvious is that recognizing the finite commitment span does not damp the incentive to run initial large primary deficits and high inflation.

IX. Are there generally useful insights from this simple, stylized, model?

This model makes strong functional form assumptions that affect its conclusions, but it embodies principles that may apply more widely. Why is it optimal to inflate early and postpone taxation? One reason is that the gap between the return on government debt and the discount rate β means that when revenue collection is postponed, the ratio of required future revenue to the revenue initially postponed grows slower than β. Its discounted present value therefore

[10]It is important that the policy optimization problem treats b_0 as freely chosen and $b_T = b^*$ as a constraint, with no connection to b_0. That $b_T = b_0$ is an equilibrium condition, not perceived as a constraint by the policy maker.

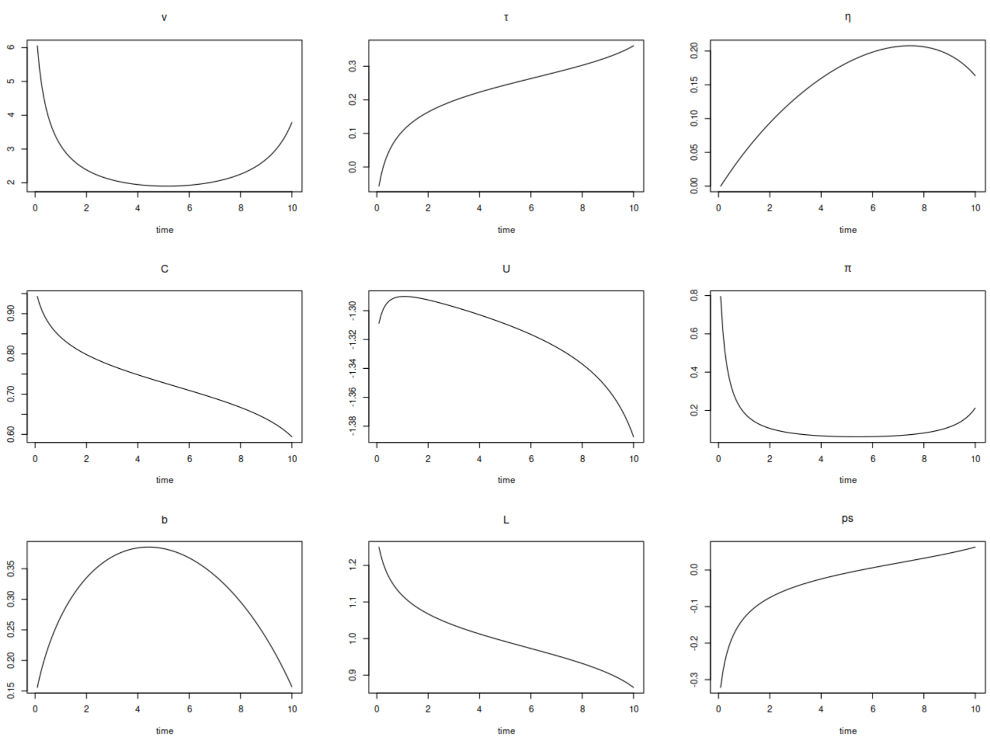

FIGURE 5. PATH WITH 10-YEAR COMMITMENT

See notes to Figure 2

shrinks the more it is postponed. The solution is not indefinite postponement, because as the required revenue grows, the costs of the distortion induced by taxation grow more than linearly with the revenue. Another reason is that high future seigniorage revenue, since it is anticipated, raises the current price level and thereby reduces the base for the current inflation tax, while high future taxes have the opposite effect on the base for the current inflation tax. These are general principles that are likely to apply in other models.

Calvo's model produces a qualitatively similar result. Though his framework, unlike this paper's, made satiation of demand for money possible in a steady state, he showed that with full commitment the economy has high initial inflation and converges to a steady state that preserves a gap between the return on money and the discount rate.

Since abandoning the full-commitment path eventually becomes extremely beneficial, it is unlikely that any government would actually follow the path for a long time. But as we have seen, a solution with temporary commitment can provide lower discounted utility than a steady-state solution. The optimal steady state solution also requires commitment, but that commitment is arguably more sustainable. The incentive to deviate from the steady state commitment is constant over time, rather than ever-growing, and it might be easier to enforce the simpler behavioral norm of constancy in tax rates.

Nonetheless, the tradeoffs the full-commitment path brings out are likely to affect policy discussion, especially in times of fiscal stress. The argument that a gap between the return on government debt and market interest rates is a fiscal resource that can be used to avoid high current levels of taxation is correct. That this will require increased fiscal effort in the future, but only a modest increase, and one that can be repeatedly postponed at modest cost, is also correct.

The paper's discussion of the full-commitment case is mainly for the $G = .25$, $\gamma = .01$ case. The qualitative results hold with $\gamma = .1$ or $G = 0$ or $G = .8$ instead. The initial inflation is smaller with $\gamma = .1$. It could of course be interesting to vary the utility function and the transactions technology. The optimal path has time varying consumption growth, and thus a time varying real rate. How would that interact with production technology if we introduced capital?

The paper's model and our discussion of it leave plenty of open questions.

X. Real world complications

Blanchard (2019), based on his Figure 15, argues that there is some evidence for a decline in the real return on capital, measured as the rate of profit relative to market capitalization. If there is such a decline[11], we could interpret it in this paper's model as a decline in θ, the rate of labor-augmenting technological improvement. This would imply that the gap between the return on debt and private asset returns might have been declining despite the stable rate of inflation, and therefore that the economy has been moving toward less reliance on seigniorage finance. This does not in itself, of course, imply that returning to previous levels of seigniorage would be optimal.

This paper uses a representative agent model, and therefore cannot consider intergenerational tax-shifting or "crowding out" issues. Blanchard and Mehrotra/Sergeyev instead assume away tax distortions in order to focus on intergenerational issues. Both aspects of debt finance are important, and should be considered jointly. Furthermore, liquidity premia on government debt vary somewhat across the term structure and across time, and currency, bearing no interest, does exist. A more serious quantitative evaluation of the effects of debt finance should consider all of these potentially important factors as operating jointly.

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THE STEADY STATE SOLUTION AND ITS UNIQUENESS

We are assuming τ constant, $b > 0$, $C > 0$, $v > 0$. The Lagrange multiplier on the agent's budget constraint is negative if $\tau > 1$, so this is ruled out by the agent's optimization. (There is no reason to work if the after-tax wage is negative.) Negative τ is in principle possible (if seigniorage is used to generate revenue that finances a labor subsidy). With $\dot{\tau} = 0$ and $\dot{b} = 0$, the system of five equations, (14), (7), (15), (16) and (17) introduced on page 12 can be solved recursively to deliver the single quadratic equation in v,

$$\gamma v^2(1 + \tau - 2G) + (\tau - G)v - \beta = 0\,. \tag{A1}$$

Its roots are

$$v = \frac{G - \tau \pm \sqrt{(G-\tau)^2 + 4\beta\gamma(1 + \tau - 2G)}}{2\gamma \cdot (1 + \tau - 2G)}\,. \tag{A2}$$

If $\tau > 2G - 1$, the equation has two real roots, one positive and one negative. Since negative v makes no sense in the model, the positive root is the relevant one. If $\tau < 2G - 1$, the roots could be imaginary, in which case they correspond to no equilibrium of the model, or they can be real and of the same sign. For the two roots to be positive, we must have $G < \tau$. But $G < \tau$ and $\tau < 2G - 1$ jointly imply $\tau > 1$, which we have noted is impossible. So the only case that delivers an equilibrium with positive v is $\tau > 2G - 1$, and in that case the steady state is unique.

IS THE STEADY STATE THE ONLY EQUILIBRIUM WITH CONSTANT τ?

The previous section shows there is only one steady state, for any τ for which a steady state exists. But could there be non-steady-state equilibria with τ constant? To check this, we allow for non-zero $\dot{b}$ in the government budget constraint (7). The other equations in the system remain unchanged from the previous section. Our derivation of the previous equation (A1) can be repeated to result in

$$-\frac{\dot{b}}{b} = \gamma v^2(1 + \tau - 2G) + (\tau - G)v - \beta\,. \tag{B1}$$

Since

$$b = \frac{C}{v} = \frac{1-\tau}{v \cdot (1 + 2\gamma v)}\,, \tag{B2}$$

b is monotonically decreasing in v, going to zero as v goes to infinity and to infinity as v goes to zero.

Proposition. *Equilibria with constant τ in which $b \to \infty$ are impossible.*

PROOF:

For an individual agent, taking the path of prices, taxes, and interest rates as given, the only state variable is b, real wealth. In any equilibrium, (4) tells us that $C < 1$ at all times, so discounted utility is bounded above. Suppose there is an equilibrium with $b \to \infty$. It can deliver no greater discounted utility than that provided by the (infeasible) allocation of $C \equiv 1$, $L \equiv 0$, which is finite. If $b \to \infty$ and therefore $v \to 0$, C converges to a positive constant, and the real rate of return on debt, $\beta - \gamma v^2$, converges to β. But if b gets large enough, spending $(\beta - \varepsilon)b$ on consumption plus transactions costs forever (where ε is some small number), while setting $L = 0$, will appear to the competitive private agent to be feasible. Transactions cost $\gamma C^2/b$ increase with increased C, but since $(\beta - \varepsilon)b$

increases linearly with C, velocity $v = b/C$ does not change as we consider higher consumption spending $C(1 + \gamma v) = (\beta - \varepsilon)b$. Thus it will appear to the private agent to be possible, as b grows without bound, to achieve a higher discounted utility than any in allocation that is actually feasible for the whole economy. This shows that an equilbrium with $b \to \infty$ does not exist. □

Proposition. *When there is a constant-tax equilibrium with b constant at $\bar{b}$, there are no constant-tax equilibria with $b > \bar{b}$.*

PROOF:

The right-hand side of (B1) goes to $-\beta$ as $v \to 0$, which implies that for large enough b, $\dot{b} > 0$. Continuity of that right-hand side, plus our result in appendix A that a constant-tax steady state, when it exists, is unique, implies that when a steady state exists, on any constant-tax equilibrium path with $b > \bar{b}$, $b \to \infty$, which is impossible, so when a steady state exists, there are no constant-tax equilibria with $b > \bar{b}$. □

Proposition. *When there is no steady state with the constant tax rate τ, there are also no constant-tax equilibria with $\dot{b} > 0$ anywhere along the entire equilibrium time path.*

PROOF:

The fact that $\dot{b}$ is positive for large enough values of b implies, with no steady state, that $\dot{b}$ is positive everywhere and thus (since there is no steady state) that $b \to \infty$, again contradicting the hypothesis that this was an equilibrium. □

Now we consider possible equilibria with constant tax rate and decreasing b. In this model we can rule out such equilibria if we can show that they imply b reaches zero in finite time. Since $b \geq 0$ is our assumption — private agents can't borrow from the government — it may seem obvious that if the equilibrium path implies b reaches zero in finite time and that $\dot{b} < 0$ at that point, the equilibrium is ruled out.

However, as can be seen from (B2), $b \to 0$ implies $v \to \infty$ and $c \to 0$, while $L \to G + (1 - \tau)/2$. In other words, as b hits zero, transactions cost absorb all of spending on consumption. At this point, an individual could stop working, and therefore have no tax obligation (if $\phi = 0$), while still having no consumption. A path converging to this point, while unpleasant, would not seem infeasible to the private agent. Given the time path of prices and interest rates, there would be no incentives to deviate from the path for the private agent.

This paradox can be avoided. For example, if $\phi > 0$, the dynamics of the model are unchanged, except that in (B1) and equations derived from it G is replaced by $G - \phi/(1-\tau)$. If the values of ϕ and G are thought of by the agent as fixed for all time, even after the agent's b is exhausted, the agent will see these paths as ones on which his tax obligations can't be satisfied. This will increase the agent's initial demand for government debt, reduce the initial price level, and thereby push the equilibrium back to the saddle path. A similar way to avoid the paradox is a trigger policy, where the tax authority promises to introduce a lump-sum tax if the economy starts on an explosive path. This leaves the saddle path with $\phi = 0$ unaffected, while eliminating the explosive paths.

To see how this works out in this model, we again rewrite the $\dot{b}$ equation, multiplying (B1) through by b and expressing everything on the right as a function of v:

$$-\dot{b} = (1-\tau)\left(\frac{\gamma v(1+\tau-2G)}{1+2\gamma v} + \frac{\tau - G}{1+2\gamma v} - \frac{\beta}{v\cdot(1+2\gamma v)}\right)\,. \tag{B3}$$

Proposition. *When τ is constant, there are no equilibria with $b \to 0$.*

PROOF:

As v goes to infinity (and b to zero), the right-hand side of this expression converges to $(1-\tau)(1+\tau-2G)/2$, a positive number if a steady state exists, implying $\dot{b}$ becomes negative and is bounded away from zero when b becomes small. But this implies that b reaches zero in finite time. Thus a path with b converging to zero cannot be an equilibrium. But we know that constant-tax steady state equilibria, when they exist, are unique, and also that $\dot{b}$ is negative for small enough b. This implies $\dot{b}$ is negative for $b < \bar{b}$ when the constant-tax steady state exists, and thus that there is no equilibrium with constant τ and $b < \bar{b}$.

When there is no steady state, i.e. $\tau + 1 < 2G$, the right-hand-side of (B3) is negative, and therefore $\dot{b}$ is positive, for large enough v (small enough b). Since the right-hand-side of (B3) is continuous, and there is no steady state, this means $\dot{b}$ must be positive for all values of b on any equilibrium path, and threfore that there are no equilibria with constant τ and b decreasing. □

This completes the argument that when τ is constant, the only competitive equilibria that exist are steady states, and that the steady state for a given constant τ is unique.

UNIQUENESS OF THE INITIAL PRICE LEVEL

The nominal government budget constraint is

$$\dot{B} = iB + GP - \tau LP\,. \tag{C1}$$

If this holds at every moment, including the initial date, it implies that B_0 is fixed and cannot be affected by policy choices. But both our optimal non-stationary and our fixed-τ, when they exist, imply an initial value for $b = B/P$ that does not depend on B. Thus our analysis implies that at time zero, the price level jumps up or down to match B/P to the equilibrium value of b. Since the equilibrium, when it exists, is unique, the initial price level is uniquely determined.

In our discussion of policy implications in the main text, we considered the possibility of an instantaneous upward jump in B. It does seem plausible that a large upward jump in B could be produced by a brief and very large transfer payment. This would involve mailing checks to the public — which was actually done during the pandemic. Such an action, if followed by a constant τ and G, would affect only the initial price level, not the subsequent real equilibrium path.

The reverse policy action, a discrete downward jump in B_0, seems less plausible. Lump sum transfers are much easier to arrange than large lump-sum taxes or wealth confiscations. In our simple representative agent model, a one-time lump-sum tax, paid for by agents selling nominal bonds, might seem possible. But with heterogeneous holdings of bonds, a uniform lump sum tax might not even be feasible because of the wealth differences, while a uniform lump sum transfer would not face such a problem.