# The Fundamental Value of a Tulip

In this note and exercise you apply the simplest version of the methods for calculating the fundamental value of an asset to the problem of valuing a tulip bulb. Unlike a stock or bond, which usually pays some return each period, a tulip bulb that is held for investment purposes has a *cost* each period. That is, to keep the bulb reproducing, you have to tend it, and this costs some resources. The only payoff from the investment comes from eventual sale of the bulbs.

If buying a bulb, planting it, and tending it in order to have more bulbs available next period is to be an attractive investment, it must pay a return comparable to other investments. Suppose the other investment is a bond that pays an interest rate $r$. The price of the bulb at $t$ is $Q_t$. If we buy a bulb and plant and care for it, the total expense required now is $Q_t$. Taking this same amount of money and investing it in a bond must yield the same return as buying the bulb and planting it. The number of bulbs available next year, per bulb planted this year, is $1+g$. We will also have to pay the cultivation cost $c$ for each bulb (which we will think of as being incurred at $t+1$). So for a bond and bulb planting to yield the same return, we must have

$$Q_t(1+r) = (1+g)Q_{t+1} - c\,. \tag{1}$$

Assume $g > r$. Then there is a unique **steady-state** value for $Q$, which we can find by setting $Q_t = Q_{t+1}$ in (1) and solving for this unchanging value of $Q$. It is

$$\bar{Q} = \frac{c}{g-r}\,. \tag{2}$$

So bulbs are expensive if they are costly to cultivate ($c$ large), but also if they are slow to reproduce ($g$ close to $r$).

But it is important to note that (2) does not tell us that $Q_t = \bar{Q}$ always. It only tells us that *if* $Q_t$ reaches the value $\bar{Q}$, it will stay there. So if, for example, $Q_t$ starts out way above $\bar{Q}$, we know from (1) that it will decline toward $\bar{Q}$ at an exponential rate. This is perhaps easiest to see if we rewrite (1) in terms of $\bar{Q}$:

$$Q_{t+1} - \bar{Q} = \frac{1+r}{1+g}(Q_t - \bar{Q})\,. \tag{3}$$

In class, and in CLM, a unique fundamental value for an asset is derived from an equation close in form to (1), because the equation in these standard situations is **unstable**. That is, it implies that there is only one value of $Q_t$ consistent with $Q_t$'s not growing exponentially forever. But in the current setup, with an asset that grows faster than the interest rate and pays a "negative dividend", Any positive initial value of $Q_t$ implies via (1) a smooth path of prices that converges to $\bar{Q}$.

How then do we decide on a fundamental value for $Q_t$? First, we must recognize that eventually some bulbs must be sold to people who are not simply cultivating them for investment purposes. If no one ever buys bulbs simply to look at them, then those people buying them for investment purposes are accumulating ever-growing wealth. The price of bulbs is declining (at roughly the rate $(1+r)/(1+g)$), at least when $Q$ is far above $\bar{Q}$, to be sure, but their number is growing at the rate $1+g$, so that the total value of the bulbs is growing at the rate $(1+r)$. It can't make sense for investors simply to hold on to this ever-growing wealth forever without spending it. This means in particular that if $\bar{Q}$ is so high that no one would ever buy a bulb at that price for non-investment purchases, there can be no sustained equilibrium in which bulbs are produced and sold for consumption at a constant price.

Suppose consumer demand for tulips is given by

$$Y_t = \begin{cases} a - bQ_t & Q_t \leq a/b \\ 0 & Q_t > a/b \end{cases}, \tag{4}$$

where $Y_t$ is the quantity of bulbs sold for consumption. If $Q_0 > a/b$, then no bulbs are sold to consumers at $t = 0$. Price will continue to drop, according to (1), until it falls below $a/b$. Call the first date at which this happens $t^*$. For $t < t^*$, the total quantity planted, $Z_t$, is growing at the rate $1+g$, because no bulbs are sold. After $t^*$, The evolution of $Z$ is governed by

$$Z_{t+1} = (1+g)(Z_t - Y_t) = (1+g)(Z_t - a + bQ_t)\,. \tag{5}$$

For the price sequence to be viable, it must imply that as $Q$ approaches $\bar{Q}$, $Z$ approaches its own steady state value, which is (from (5))

$$\bar{Z} = \bar{Y}\frac{1+g}{g} = (a - b\bar{Q})\frac{1+g}{g}\,. \tag{6}$$

At this level of $Z$, the new growth each year is just enough to satisfy consumption demand at the steady-state price, so the amount planted remains constant.

If we start from some arbitrary initial $Q_0$, the implication of (1) and (5) together will almost certainly be either that $Z$ grows larger without bound—which is not possible because it implies irrational behavior—or that $Z$ eventually becomes negative, which is physically impossible. There is only one value of $Q_0$ that is consistent with $Z$ approaching its own steady state as $Q$ approaches its steady state. This unique $Q$ is the fundamental value of a tulip. However, as should be clear from the foregoing discussion, the fundamental value at a given time depends on what the current stock of tulips $Z_t$ is. With very high initial $Z$, initial $Q$ will be low, possibly even below $\bar{Q}$. With very low initial $Z$, initial $Q$ will be high, possibly even above $a/b$.

A situation in which $Q_t$ is on a path that is above its fundamental value but still satisfies (1) at every $t$ constitutes what is known as a **rational bubble**. Everyone who actually invests in the bulbs and sells them is making a reasonable return on investment. But in this situation, as in any rational bubble, there must be someone who is holding ever-growing amounts of wealth without ever spending it. With an asset like a stock, paying a positive dividend, a bubble implies a steadily rising asset price.

For our version of a tulip, the price (if it starts out higher than $\bar{Q}$) declines even if there is a bubble. The question of whether a bubble exist has entirely to do with how $Z$ is behaving. It is interesting that neither Garber nor Kindleberger brings out this point. If there were a truly persistent rational bubble in the Tulip Mania, the earth would probably be covered to a depth of many miles with tulips by now.

Investors who cannot foresee the future, or who do not know the demand curve for a particular bulb variety, can make big mistakes without being irrational. If they overestimate demand, they may pay too high a price for bulbs when there are only a few available. Once the bulbs start being sold, if the demand is lower than expected, the price may suddenly drop to reflect the new information (or suddenly rise if the demand is better than expected). Of course you may—and this seems to be Kindleberger's view of most bubbles and panics—argue that any sensible person should have been able to see that prices before the price drop were above the fundamental value. In the tulip case, this would be the argument that only values of $a$ and $b$ that could easily be seen to be impossible would be consistent with the observed $Q_t$ at the peak of the boom in tulip prices.

**Exercise due 9/28**

Suppose the demand curve for tulips is of the form (4), with $a = 1000$, $b = 200$, and that in (1) we have $g = 1.5$, $r = .1$, and $c = 1$. The initial stock of bulbs available is $Z_0 = 2$.

a. Find $\bar{Q}$, the steady-state price, $\bar{Y}$, the steady-state quantity consumed, and $\bar{Z}$, the steady-state quantity planted for investment. You will see that $Z_0 < \bar{Z}$, which implies (be sure you can explain why) that if $Q_0$ reflects fundamentals, $Q_0 > \bar{Q}$.

b. Suppose $Q_0 = 10\bar{Q}$. Determine whether this initial price is above or below the fundamental value. [Use (1) and (5) to find the values of $Z_1$ and $Q_1$ from the given values of $Z_0$ and $Q_0$, $Z_2$ and $Q_2$ from your computed values of $Z_1$ and $Q_1$, etc. If $Q_0 = 10\bar{Q}$ is too low, you will find $Z$ turning negative after $Q$ gets close to $\bar{Q}$, whereas if $Q_0$ is too low, you will find $Z$ exploding upward after $Q$ gets close to $\bar{Q}$. This can all be done with a calculator, but it will be easier if you make good use of a spreadsheet program.

c. Repeat your analysis in (b) for the case where $Z_0 = 2000$ and $Q_0 = .5\bar{Q}$.

d. (Not required) You might be interested in trying to determine what the fundamental value actually is in (b). You can estimate it by trying values of $Q_0$ that are too high and too low, continually narrowing the interval until you've reached the accuracy you want. This would be very tedious if you are doing everything with a calculator, but if you have done (b) and (c) with a computer, this is scarcely any additional work.