Final Take Home Exam

1. FTPL

Consider a model in which the real rate of interest is constant at ρ (either because consumption is constant or because there is an asset with a fixed rate of return—we don't model the reason). The government begins at time $t = 0$ with a policy of pegging the price level P at a constant $\bar{P}$. It also at the start has a fiscal policy that makes the level of per capita lump sum taxes τ respond to the level of per capita real government debt according to

$$\tau = -\phi_0 + \phi_1 b\,. \tag{1}$$

However, there is a maximum feasible level of taxation $\tau = \bar{\tau}$. If taxation ever reaches this level, the government abandons the fixed-P policy in favor of a fixed-r policy that sets $r = \rho$, while setting $\tau \equiv \bar{\tau}$ forever thereafter.

The government's budget constraint is

$$\dot{B} = rB - P\tau\,, \tag{2}$$

the usual definition of the real rate applies, i.e.

$$\rho = r - \frac{\dot{\hat{P}}}{P}\,, \tag{3}$$

and we are using our usual notation for real and nominal debt,

$$b = \frac{B}{P}\,. \tag{4}$$

The solution to this model will have the government following (1) and the price peg for some initial period of time $(0, T)$, then switching to the $r = \rho$, $\tau = \bar{\tau}$ policy. It is possible that T turns out to be 0, i.e. that the attempt to fix P cannot be sustained even for an instant.

i. Explain why the price level cannot jump discontinuously at $T > 0$ if agents are willingly holding positive, finite amounts of government debt.
ii. Find the constant level $\bar{b}$ of b that prevails for dates $t \geq T$, as a function of $\bar{\tau}$ and ρ.
iii. Use your analysis in i and ii to find an expression for B_T as a function of $\bar{\tau}$, ρ, and $\bar{P}$.
iv. Find a differential equation for B that applies for $t \in (0, T)$, making $\dot{B}_t$ a function of B_0, ρ, ϕ_0, ϕ_1, $\bar{P}$, and t.
v. Using your results in iii and iv, write down an equation that equates two expressions for B_T and can be solved for T. Say what you can about how changes in ϕ_0, ϕ_1, and $\bar{P}$ affect how long the initial regime is sustainable, Is it possible that $T = \infty$,

i.e. that the original policy regime can be sustained forever? Could the parameters be set so that $T = 0$?

vi. Is it possible that τ jumps discontinuously at T? Explain your answer.

vii. Describe how the bond market would be likely to react if the government attempted to persist with its fixed-P policy beyond T, and explain why this forces the government to abandon the policy at T.

2. Borrowing When (Nearly) Bankrupt

Consider a firm that has available an investment process in which by investing S this period the firm receives a gross return of Sz next period. The gross rate of return z is random, distributed as $N(1.05, .09)$ (Normal with mean 1.05, variance .09). The firm can borrow at the gross interest rate R. If it borrows B and invests S this period, its return next period is $Sz - RB$—unless it goes bankrupt. It goes bankrupt when $Sz - RB < 0$, in which case the firm's gross return is simply zero, not negative, and the lender only retrieves Sz, not the promised RB.

i. Draw, sketch, or construct with a computer a plot of the expected rate of return to the firm as a function of B/N, assuming the firm's budget constraint sets $S = B+N$ in the first period. Here N stands for the firm's net worth, which the firm takes as given. You will need to find the expectation of a "truncated normal" distribution. If X is $N(0, \sigma^2)$, and if $Y = X$ whenever $X > a$, but $Y = a$ if $X \leq a$, then $E[Y] = \Phi(a/\sigma) \cdot a + \sigma\phi(a/\sigma)$, where $\Phi(\cdot)$ is the cdf of a standard normal (N(0,1)) distribution and $\phi(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ is the pdf of the standard normal. Both these functions can be looked up in tables or evaluated in most spreadsheets.

ii. Comment on what your plot suggests about the effects on investment decisions of low net worth.

3. Martingales and Brownian Motion

i. Suppose an asset price process $Q(t)$ behaves as follows. For s between any successive integer dates t and $t+1$ $Q(s)$ follows a straight line with random slope $\delta(t)$, which remains constant over the interval $(t, t+1)$. The $\delta(t)$'s are independent of one another and all normally distributed with mean zero and variance one. Is this a martingale process? Is it a Brownian motion? Does it provide an opportunity for an investor who can always borrow or lend at the same fixed interest rate r and invest in the security with price Q to make risk free returns at rate higher than r?

ii. Answer the same three questions for the case where Q instead behaves as follows: Q is constant, except at randomly timed moments at which it jumps discontinuously. The probability of a jump in any interval is uniform through time, and independent across non-overlapping time intervals. If a jump occurs, its probability distribution is $N(.2Q(t), .09Q(t)^2)$