Answers to Take Home Final Problem Set

1. FTPL

(i) A discontinuous jump (up or down) in prices imposes a discontinuous capital loss or gain on holders of dollar-denominated government liabilities. The problem statement says the real rate of interest is constant, however, either because of constant consumption or because there is an asset with a fixed rate of return. If there is an asset other than government bonds that has a fixed rate of return, the occurrence of an anticipated upward or downward jump in the price level creates an arbitrage opportunity. If it is an upward jump in prices, bond holders can avoid the loss by exchanging their bonds for the other asset an instant before the price jump, then buying them (plus some more, because they are now cheaper) an instant after. They would reverse this strategy if the jump was to be downward. Their doing this will create price pressures in the market that prevent a price jump.

If there are no other assets than government liabilities, a correct argument is a little more subtle. It is still true that there is a discontinuous capital loss or gain on nominal government liabilities if price jumps, but it is not as clear what a bond holder can do about it. In this model, all that can be done is to increase consumption just before an upward jump and decrease it after—in other words use up wealth faster before its value deteriorates with the jump. If, say, the price level increases by a factor of 2 at some known date t_0, then an optimizing agent will consume faster before t_0 than after, so that the marginal utility of wealth just after t_0 is twice what it is just before t_0. This is the condition under which shifting consumption from just after to just before t_0 no longer produces gains in utility. But the problem statement says that either there is an asset with a fixed real rate of return or consumption is constant. With constant consumption and no other asset optimizing agents will not be satisfied to hold bonds while the price level jumps.

(ii) After T, we are in the simple situation we discussed in class and in the exercise. Government debt satisfies the equation

$$\dot{b} = \rho b - \bar{\tau}\,. \tag{A1}$$

This equation has only one stable solution,

$$b \equiv \bar{b} = \frac{\bar{\tau}}{\rho}\,. \tag{A2}$$

Unstable solutions can be ruled out by the argument that b exploding upward while C remains bounded cannot be optimal, while b exploding downward is inconsistent with bounds on borrowing from the government by private agents.

(iii) Since there is no discontinuity in prices at T, we have

$$\frac{B_T}{P_T} = \frac{B_T}{\bar{P}} = \bar{b} = \frac{\bar{\tau}}{\rho}\,, \tag{A3}$$

or

$$B_T = \frac{\bar{\tau}\bar{P}}{\rho}\,. \tag{A4}$$

(iv) The equation is just the government budget constraint ((2) in the exam text), with equation (1) substituted in for τ and ρ substituted for r (based on (3) in the exam text and the fact that P is being held constant). The result is

$$\dot{B} = \rho B - (-\phi_0 + \phi_1 b)\bar{P} = (\rho - \phi_1)B + \phi_0\bar{P}\,. \tag{A5}$$

(v)

(vi)

(vii) Equation (A5) is a first-order linear differential equation with fixed coefficients, which we know how to solve. The solution will be in the form

$$B_t = \kappa_1 e^{(\rho-\phi_1)t} + \kappa_0 \tag{A6}$$

with

$$\kappa_0 = \frac{\phi_0\bar{P}}{\phi_1 - \rho}\,, \quad \kappa_1 = B_0 - \kappa_0. \tag{A7}$$

We can immediately convert this into an expression for B_T:

$$B_T = \left(B_0 - \frac{\phi_0\bar{P}}{\phi_1 - \rho}\right) e^{(\rho-\phi_1)T} + b^*\bar{P}\,, \tag{A8}$$

where $b^* = \phi_0/(\phi_1 - \rho)$ is the steady-state value of b in (A5), i.e. the steady-state b on the assumption that the initial fiscal policy stays in place.

But we already have a different expression for B_T in (A4). Equating the right-hand-sides of our two expressions for B_T gives us an equation that involves only constants, and T. That equation can be arranged to read

$$e^{(\rho-\phi_1)T} = \frac{\frac{\bar{\tau}}{\rho} - b^*}{\frac{B_0}{\bar{P}} - b^*}\,, \tag{A9}$$

which implies

$$T = \frac{1}{\rho - \phi_1}\log\left(\frac{\frac{\bar{\tau}}{\rho} - b^*}{\frac{B_0}{\bar{P}} - b^*}\right)\,. \tag{A10}$$

We need to consider a number of possible cases here. If there is to be a switch in policy, T from (A10) must turn out to be a finite positive number.

This means first that the numerator and denominator of the argument of the log function on the right hand side must have the same sign. Then in addition the right-hand-side must turn out to be positive. So a positive, finite T requires one of the following sets of conditions:

$$\rho > \phi_1 \quad \text{and} \begin{cases} \dfrac{\bar{\tau}}{\rho} > \dfrac{B_0}{\bar{P}} > b^* \\ \text{or} \\ b^* > \dfrac{B_0}{\bar{P}} > \dfrac{\bar{\tau}}{\rho} \end{cases} \tag{A11}$$

or

$$\phi_1 > \rho \quad \text{and} \begin{cases} \dfrac{B_0}{\bar{P}} > \dfrac{\bar{\tau}}{\rho} > b^* \\ \text{or} \\ b^* > \dfrac{\bar{\tau}}{\rho} > \dfrac{B_0}{\bar{P}} \end{cases} \quad . \tag{A12}$$

Just before T, if it is a positive, finite number, we will have, because of the fiscal policy equation (1) in the exam text, $\tau = \tau^* = -\phi_0 + \phi_1 \bar{b}$. This value of τ is not in general going to match $\bar{\tau}$. If $\tau^* > \bar{\tau}$, then the condition that triggers the policy switch, i.e. $\tau \geq \bar{\tau}$, will have occurred before T, which contradicts our solution. In this case there is no solution for non-zero T. Every potential solution that involves positive T fails to have a big enough stream of taxes to back the initial level of real debt at $B_0/\bar{P}$. The second line of (A11) and the first line of (A12) both imply that $\tau_T > \bar{\tau}$, and thus both actually imply $T = 0$.

If $\bar{\tau} = \tau^*$, then the switch occurs at T with no discontinuity in τ.

If $\bar{\tau} > \tau^*$, then the switch occurs at T and τ jumps upward to $\bar{\tau}$ at T. This jump in τ, even though it has been stated that the level of taxes jumps when τ hits $\bar{\tau}$, is paradoxical, and the problem's statement should have made it clear that this could occur, instead of hinting at it in the questions asked. What happens is that people in this economy, realizing what the trigger level of taxes is, see that in order to make the stream of taxes have a sufficient discounted present value to back the initial real debt at the value $B_0/\bar{P}$, τ must jump to $\bar{\tau}$ before the original fiscal policy brings τ up to that level. If the fiscal authority attempted to persist with the fiscal policy (1) after T, the public would attempt to unload its debt, which it now would perceive as insufficiently backed. Or to put the same thing another way, the public would perceive that at the value B_T/P_T, the debt they hold gives them enough wealth to spend more than their after-tax income indefinitely, so they would attempt to do so. The result would be pressure on P to increase above $\bar{P}$ so strong that the government could not resist it and would instead have to switch immediately to the $\tau \equiv \bar{\tau}$ policy.

The problem's somewhat unclear statement might have been interpreted to mean that the fiscal policy would *never* switch except when $\tau = \bar{\tau}$. Under this interpretation, there could never be a positive, finite T except when $b^* = \bar{b}$— but (A10) tells us that in this case there is no positive, finite solution for T. But if we maintain the interpretation that when future taxes are too low, the government is forced to abandon its attempt to peg P and to immediately switch to the fixed-τ policy, it is hard to see why they couldn't in essentially the same way be forced to switch policies discontinuously at a later date T.

When there is no positive, finite T, two cases are possible. One is that $T = \infty$, so that the fixed-P equilibrium is sustainable forever. This occurs if $\phi_1 > \rho$, $b^* < \bar{b}$ and $B_0/\bar{P} < \bar{b}$. Then debt converges smoothly to b^* without ever pushing τ above the trigger level. The other is that only $T = 0$, i.e. immediate collapse of the price-fixing, is a solution. This occurs in the cases already discussed that have finite T but imply $\tau_T > \bar{\tau}$. It also occurs in any other case when $-\phi_0 + \phi_1 B_0/\bar{P} > \bar{\tau}$, as this implies immediate triggering of the switch in fiscal policy.

It was not practical to discuss every possible case (and you were not asked to). A reasonable summary of the conclusions is that when there is a well-defined, positive, finite T, corresponding to the first line of (A11) or the second line of (A12), increasing $\bar{\tau}$ or decreasing b^* increases T. In such cases moving ϕ_1 closer to ρ also increases T. And both $T = \infty$ and $T = 0$ are possible.

2. Borrowing When (Nearly) Bankrupt

On this problem, I forgot to give you a numerical value of R, the gross interest rate and N, firm net worth. I'll display results assuming $R = 1.03$ and $N = 1$. The firm's return is $\max((N + B)z - RB, 0) = \max((1 + B)z - 1.03B, 0)$. This is the maximum of a $N(1.05 + .02B, .09(1 + B)^2)$ random variable and 0. In the problem statement you were told how to find the expectation of the maximum of a $N(0, \sigma^2)$ random variable and some truncation level a. To convert to this case, we describe the firm's return equivalently as $1.05 + .02B$ plus the max of a $N(0, .09(1 + B)^2)$ random variable and $-1.05 - .02B$. We can plot the expected return as a function of B as in Figure 1. The plot has a simple shape, so hand computation of just a few points would have been enough to give a good sketch. The message of the graph is that return is increasing in leverage, and that it increases at an increasing rate. Low net worth therefore makes a borrower in this situation especially eager to borrow. (What follows was not asked for in the question.) Lenders earn $\min(RB, (B + N)z)$. Their expected total return is $1.05(B + N)$ minus the expected return of the firms. The expected gross rate of return for lenders is therefore never higher than 1.03 and falls below 1.0 when leverage reaches about 2.4 times net worth. So lenders must limit the amount of borrowing by low-net-worth firms.

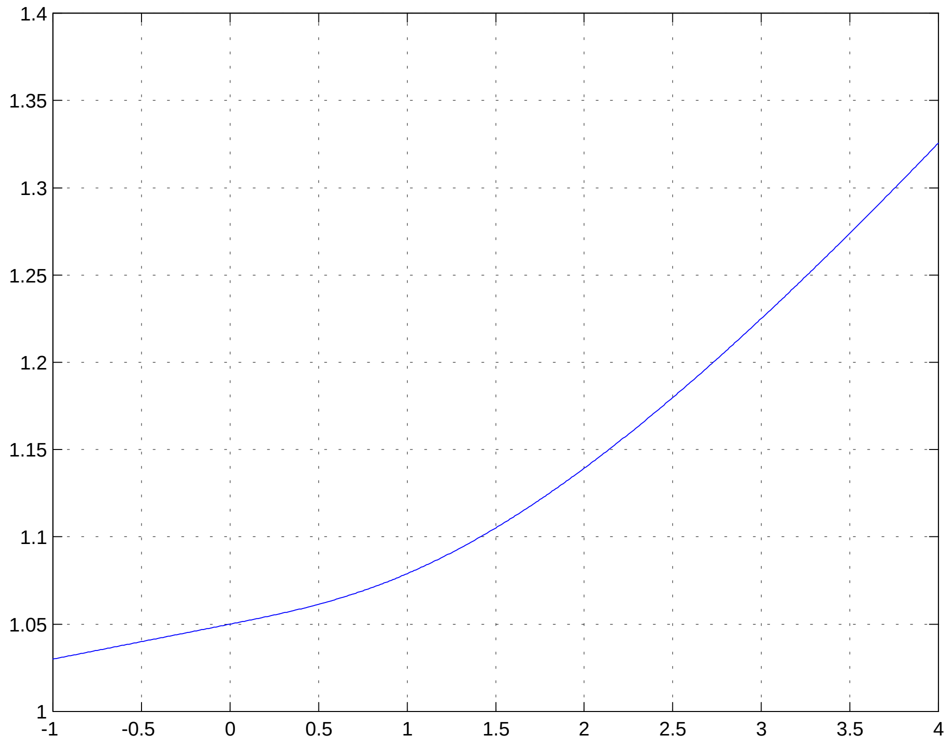

FIGURE 1. Expected Gross Rate of Return as a Function of Leverage

3. MARTINGALES AND BROWNIAN MOTION

(i) This is not a martingale. A martingale has $E_t[Q(t+s)] = Q(t)$ for all t and all $s \geq 0$. For this process this holds at integer-values of t and s, but not at non-integer t. If we write $[t]$ to represent the largest integer less than or equal to t, then for non-integer t and $s > [t]+1-t$, $E_t[Q(t+s)] = Q(t)+\delta\cdot([t]+1-t)$, for example. Q is also not a Brownian Motion, since a Brownian Motion is a special case of a martingale.

Since the asset appreciates or depreciates at a fixed linear rate between every pair of integers, the following strategy delivers a rate of return higher than r. Just after each integer date t, determine whether the slope δ satisfies

$$\frac{\delta}{Q(t)} > e^r - 1 \,. \tag{A13}$$

If so, borrow a large amount at the rate r and invest it in the asset, holding the asset until $t+1$. Because the slope of $Q(t)$ is constant over the time interval, there is no risk of any loss from this strategy, and the rate of return can be as high as desired, the higher the greater the amount of money borrowed relative to net worth. If instead the inequality in (A13) is reversed, sell the asset and lend the proceeds. If short selling is allowed, unbounded risk free rates of return are available from this strategy also. If short selling of the asset is not allowed and the investor has low net worth, high rates of return may be available only in those periods when (A13) is satisfied. But since this will certainly occur, and when it does the rate of return is unbounded, the need to wait for (A13) to be satisfied does not limit the attainable expected rate of return.

(ii) This also is not a martingale. To see this, consider $E_t[Q(t+s)]$. The random variable $Q(t+s)$ will be equal to $Q(t)$ if there is no jump between t and $t+s$, and otherwise it will be $Q(t)$ plus one or more jumps. Since the jumps all have positive expectation, it must be that $E_t[Q(t+s)] > Q(t)$, violating the defining property of of a martingale. And of course again this means it is also not a Brownian Motion. However, this process presents no opportunity for a risk free gain. Over any time interval in which an investor holds a non-zero position in this asset, there is a risk that there will be a jump. Since the jump is normally distributed, there is always some probability of a large negative jump when the position held is positive, and of a large positive jump when the position held is negative (short). No pattern of borrowing or lending, therefore, can create a risk-free gain at greater than the risk-free rate of interest.