Linear Algebra: The Bare Bones

In this course we do not make use of advanced results from linear algebra, but matrix notation and the the ideas of matrix multiplication and inversion will be useful to us. These notes present everything we will need, though you may want to consult textbooks for more explanation.

1. Basic Matrix Notation

An $m \times n$ **matrix** A is a collection of numbers, arranged in rectangular table with m rows and n columns. We often write

$$\underset{m \times n}{A} = [a_{ij}] \tag{1}$$

to indicate that A is a matrix with "typical element", in the i'th row and j'th column, a_{ij}. For example,

$$\underset{3 \times 2}{A} = \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 0 \end{bmatrix} = [a_{ij}] \tag{2}$$

is a 3×2 matrix, in which $a_{21} = 4$. An $n \times 1$ matrix is called a **column vector** and a $1 \times n$ matrix is called a **row vector**. If $A = [a_{ij}]$ is an $m \times n$ matrix with typical element a_{ij}, then the **transpose** of A, is the $n \times m$ matrix

$$A' = [a_{ji}] \ . \tag{3}$$

2. Matrix Product and Inverse

The product of two matrices is defined by

$$\underset{m \times n}{A} \, \underset{n \times p}{B} = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right] \ . \tag{4}$$

The notation on the right of (4) stands for a matrix whose i'th row, j'th column element is given by the expression in brackets. The product is $m \times p$. Notice that the number of columns in A matches the number of rows in B in (4). If they did not match, the product would not be defined.

Matrix multiplication is different from ordinary multiplication in that it is not commutative—$AB \neq BA$. In fact, unless $m = p$ in (4), the product BA is not even defined.

Notice that 1×1 matrices behave just like ordinary numbers. That is, the definition of a matrix product becomes just ordinary multiplication when A and B are both 1×1.

The **identity** matrix is the square (i.e., with matching numbers of rows and columns) matrix

$$\underset{n\times n}{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} . \tag{5}$$

It has the property that for any $m \times n$ matrix A,

$$\underset{m\times m}{I} A = A \underset{n\times n}{I} = A . \tag{6}$$

The **inverse** of a square matrix A is a matrix C with the property that $CA = I$. If C has this property, then we write $C = A^{-1}$. Though it is possible to define something like an inverse for a non-square matrix, we won't need to consider how to do so. We will think of non-square matrices as having no inverses. Some square matrices also have no inverses. We will see shortly how to check whether an inverse exists. If an inverse does exist, it is unique—that is, the problem of finding A^{-1} either has no answer, or it has a single answer. Furthermore, despite the fact that matrix multiplication is not in general commutative, left and right inverse matrices are the same. That is,

$$A^{-1}A = AA^{-1} = I . \tag{7}$$

3. Computing Inverses; Determinants

For a 1×1 matrix A, the determinant $|A|$ is the single element of the matrix.

For an $n \times n$ matrix A with $n > 1$, the ij'th **cofactor** $|A|_{ij}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A, multiplied by $(-1)^{i+j}$. For a 2×2 matrix, therefore, the cofactors are all determinants of 1×1 matrices, i.e. $|A|_{11} = a_{22}$, $|A|_{12} = -a_{21}$, etc.

It turns out that for any $n \times n$ matrix A,

$$|A| = \sum_{j=1}^{n} a_{ij} \, |A|_{ji} \ . \tag{8}$$

It does not matter how we pick i in (8). Every i will yield the same answer for the determinant of A. Applying (8) to the $n = 2$ case, we get

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \, . \tag{9}$$

Since this tells us how to compute the determinant of a 2×2 matrix, and since all the cofactors of a 3×3 are themselves 2×2, we can apply (8) to find the determinant of any 3×3. And knowing how to form determinants of 3×3's allows us to apply (8) to find the determinant of any 4×4, etc.

A formula for the inverse of a matrix that works for hand calculation on small matrices (3×3 or smaller, or for some larger matrices that are mostly made up of zeros) is

$$A^{-1} = \left[\frac{|A|_{ji}}{|A|} \right] . \tag{10}$$

For example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 1 \cdot 4 - 2 \cdot 3 = -2 \,, \quad A^{-1} = \begin{bmatrix} \frac{4}{-2} & \frac{-2}{-2} \\ \frac{-3}{-2} & \frac{1}{-2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix} . \tag{11}$$

No matter what size matrix you are working with, or what method you use, it is always good to verify your calculated inverse by multiplying out $A^{-1}A$ to be sure that it is in fact I.

Because (10) involves division by $|A|$, it does not work for matrices with $|A| = 0$. Such matrices have no inverses, and are called **singular** matrices.

4. Useful Facts about Matrices and Determinants

(4.a) $|A| = |A'|$.
(4.b) If any row or column of A consists entirely of zeros, $|A| = 0$.
(4.c) If any two rows or columns of A are identical, $|A| = 0$.
(4.d) $|A| = 0$ if and only if there is a row vector x that does not consist entirely of zeros ($x \neq 0$) such that $xA = 0$.
(4.e) $|A| = 0$ if and only if there is a column vector x that does not consist entirely of zeros ($x \neq 0$) such that $Ax = 0$.
(4.f) If we **partition** the $(m+n) \times (m+n)$ matrix A, i.e. write it as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \tag{12}$$

with A_{11} $m \times m$, and if $A_{12} = A_{21} = 0$, then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} . \tag{13}$$

5. Practice Exercises

The exercises in this section are not required work for the course. They are meant to make you use the material in the notes to test and strengthen your understanding of it. If you want to have feedback on this, you can submit answers to the exercises for grading, or ask questions about the exercises in class or office hours.

(5.a) Prove 4.b directly from the definition of a determinant in (8).
(5.b) Prove that 4.c follows from 4.d and 4.e.

(5.c) Compute the inverse of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} .$$

(This is a surprising amount of work, even with a calculator. You should do it so that you understand how useful it is to avoid having to do this by hand, even with a calculator. Be sure to check your answer.)

(5.d) Show that if we change the lower right element of the matrix in 5.c to 9, we get a singular matrix, i.e. show that

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 .$$

It is much easier to do this by applying 4.d or 4.e than by grinding ahead with calculating the determinant.