Model-Solving Exercise, due 2/19

In this exercise you will find an approximate solution to the linear-quadratic permanent income model with inequality constraints. The problem or a close relative has been given as a linear model-solving problem in this and previous year's 511b classes. But the exact solution requires numerical methods. An exercise and an answer sheet discussing a version of the problem are posted on the web page for 511b.

All versions of the model we consider have the objective

$$\max_{\{C_t, A_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \left(C_t - \tfrac{1}{2} C_t^2\right)\right] \tag{1}$$

subject to

$$A_t \le (1+r) A_{t-1} + Y_t - C_t \tag{2}$$

$$A_t \ge 0 \ . \tag{3}$$

i) To keep the problem as simple as possible to start with, we begin with the case where

$$\beta = (1+r)^{-1} = .95 \tag{4}$$

$$Y_t \sim N(.8, .01), \text{ i.i.d. across } t. \tag{5}$$

Find the first order conditions. Note that because of the linearity, parameterizing the expectation in the FOC and parameterizing the decision rule are essentially equivalent here. Form a parametric family of decision rules as

$$C_t = \min\left\{1, \gamma_0 + \gamma_1 \cdot \left((1+r)A_{t-1} + Y_t\right) + \gamma_2 \cdot \left((1+r)A_{t-1} + Y_t\right)^2\right\} \ . \tag{6}$$

Note that we know that C can never exceed 1, by the basic logic of the model as laid out in the 511b answer sheet. It also is not hard to verify that when we drop (3) and impose a requirement that the solution not explode exponentially, we get a linear decision rule of the form of the right-hand term in brackets in (6), with $\gamma_2 = 0$. A good initial value for the vector of γ's would be the optimal γ_0 and γ_1 from this linear variant on the model. You may alter the parameterization, so long as the space of functions is not altered. So for example you could make γ_1 and γ_2 multiply deviations of $W_t = (1+r)A_{t-1} + Y_t$ and W_t^2 from guesses of their means, or more sophisticated tricks to make the variables that different γ's multiply less collinear. This may even be necessary to get convergence.

i.a) Find the γ vector that makes the cross products of the FOC residual vector with a constant vector, $\{W_t\} = \{(1+r)A_{t-1} + Y_t\}$, and $\{W_t^2\}$ all 0. You can use `csolve`, a matlab routine posted on my web page, or any other equation solver. Use repeated simulations of length 50 years, starting from an initial $A_{-1} = .5$. 200 repetitions should be enough, at least to start. Plot the resulting $C(W)$ function.

i.b) Add a cubic term and see if it makes a difference.

i.c) Try (i.a) with Marcet's method of iteration, where instead of using an equation-solver, one does non-linear least squares (which here would reduce to linear least squares) at each iteration.

ii) Repeat the exercise for the case where (5) is replaced by

$$Y_t = .2 + .8Y_{t-1} + \varepsilon_t\,,\ \ \varepsilon_t \sim N(0, .01)\,. \tag{7}$$

Note that this will require letting the decision rule depend separately on Y and A, as W is no longer by itself a complete description of the state. Even a general quadratic formulation here will involve 7 free coefficients. You should probably begin by allowing only the "own" quadratic terms in Y^2 and A^2 to enter, suppressing the cross terms. Shrewdness in adjusting the parameterization to avoid collinearity will be more important here. Draw the initial Y from the steady-state marginal distribution implied by (7).

Remember, in all of this: The pseudorandom numbers you will draw here for Y, or for ε in (ii), are drawn just once, before you begin your iterations. You can then check your solution by drawing a new set of exogenous random terms and verifying that your solution does not change much if you use the new draws.