EXERCISE ON CONTINUOUS TIME FTPL AND RATIONAL INATTENTION

(1) **Balance sheets and inflation control** Consider an economy in which the representative agent solves

$$\max_{C,M,B} E\left[\int_0^\infty e^{-\beta t}\log(C_t)\,dt\right] \tag{1}$$

subject to

$$C_t\cdot(1+\gamma v)+\frac{\dot{B}+\dot{M}}{P}=Y+\frac{rB}{P}-\tau\,. \tag{2}$$

Here $v = PC/M$. The central bank has the budget constraint

$$\frac{\dot{M}-\dot{B}_C}{P}=\frac{-rB_C}{P}+\tau_C\,. \tag{3}$$

τ_C is the rate (in real terms) at which the bank makes transfers to the fiscal authority. It is natural to suppose that it is positive, enough to keep the bank's excess of assets over liabilities small, when there is net positive seignorage. The treasury has the budget constraint

$$\frac{\dot{B}_F}{P}+\tau+\tau_C=\frac{rB_F}{P}\,. \tag{4}$$

Market clearing requires that debt issued by the treasury be the sum of debt held by the central bank and debt held by the public, i.e.

$$B_F = B_C + B\,. \tag{5}$$

(a) Suppose the monetary policy has been to make M grow at a constant rate of $-\frac{1}{2}\%$ per year, that this policy has been anticipated by the public and expected to go on forever. Show that, if we assume constant Y and accommodating fiscal policy, the economy has a steady-state equilibrium with constant r, C, and $\pi = \dot{P}/P$.

Agents' FOC's are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t(1 + 2\gamma v_t)$$

$$\partial B: \qquad -\frac{\hat{d}}{dt}\left(\frac{\lambda}{P}\right) + \beta\frac{\lambda}{P} = \frac{r\lambda}{P}$$

$$\partial M: \qquad -\frac{\hat{d}}{dt}\left(\frac{\lambda}{P}\right) + \beta\frac{\lambda}{P} = \frac{\lambda\gamma v^2}{P}$$

The $\frac{\hat{d}}{dt}$ notation stands for "right derivative of the expected time path". I.e., $\frac{\hat{d}x(t)}{dt} = \lim_{s\to 0} E_t[x(t+s) - x(t)]/s$. Because we've assumed that Y is constant, there is no source of exogenous randomness in the model, and the $\hat{d}/dt$ notation is just a right-derivative. The B and M FOC's together imply the liquidity preference relation $r = \gamma v^2$. Using all three budget constraints together, we get the social resource constraint $C \cdot (1 + \gamma v) = Y$, so the constant v (and Y) imply constant C. This means that P and M must be proportional, and thus that $\pi = \hat{\dot{M}}/M = .005$. Therefore, from the bond FOC, $r = \rho + .005$. If use this r in the liquidity preference relation to determine v, we will then have satisfied both the ∂B and ∂M FOC's. We can determine C from the SRC. To satisfy the central bank and treasury constraints without violating transversality, we can assume, say, that the bank picks τ_C so as to keep B_C constant, and then that the treasury sets $\tau = \frac{rB_F}{P} - \tau_C$, so that B_F/P remains constant.

(b) Show that with fiscal policy setting $\tau + \tau_C = 2\beta B/P$, this equilibrium is not unique. There are also equilibria in which v explodes to infinity (but none in which it goes to zero).

Subtracting the treasury and central bank budget constraints from the private budget constraint, we get the social resource constraint

$$C(1 + \gamma v) = Y\,. \tag{†}$$

Note that we can, using the ∂C FOC, define

$$Z = M\frac{\lambda}{P} = \frac{1}{v(1 + 2\gamma v)}\,. \tag{$**$}$$

Then the ∂M FOC becomes

$$-\frac{\dot{Z}}{Z} = \gamma v^2 - \beta - g_m\,, \tag{$*$}$$

where g_m is the growth rate of M. Because Z is monotone decreasing in v, and goes to $\pm\infty$ as v goes to $\mp\infty$, this is an unstable differential equation.

If v is above its steady-state value, Z decreases toward zero at an ever increasing rate. If v is below its steady-state value, Z increases at a rate that approaches $\beta + \gamma$.

In equilibrium it is impossible for v to go to zero, because this would entail arbitrarily large M/P while C remains bounded above by Y. Because transaction costs become trivially small for small enough v, the effect on discounted future transactions costs from consuming a given proportion of real balances becomes arbitrarily small as $v \to 0$, while the current utility gain from consuming the given fraction goes to infinity has $M/P \to \infty$.

But we can't rule out $v \to \infty$ here. No transversality condition is violated by the behavior of M/P in this case, and there is no problem with feasibility.

So far, though, we have not considered the behavior of real government debt on these time paths for P and v. The problem statement was, unfortunately, incomplete here. I was thinking that τ_C was the same as seignorage revenue, but nothing in the problem statement requires that this be true. The central bank could be running down its net worth by large transfers to the treasury every period, or it could be setting τ_C to zero and accumulating ever-increasing net worth. A natural way to complete the problem statement is to assume that all monetary expansion is financed by open market operations, i.e. that $\dot{M} = \dot{B}_C$, which implies $\tau_C = rB_C/M$. Using this additional assumption, plus the $\tau + \tau_C = 2\beta B/P$, and introducing the notation $b = B/P$ for real debt and $m = M/P$ for real money balances, we can rewrite the budget constraint as

$$\dot{b} + \dot{m} = \beta(b+m) - 2\beta b = -\beta(b+m) + 2\beta m\,.$$

On an equilibrium path in which v is exploding upward, m is determined by equation ($*$) and the social resource constraint (†). The government budget constraint is then a stable differential equation in $b + m$, because the coefficient on the level of $b + m$ is negative, driven by the forcing process m, which is tending to zero. The explicit solution is

$$b_t + m_t = 2\beta \int_0^t e^{-\beta s} m_{t-s} ds + (b_0 + m_0) e^{-\beta t}.$$

This will go to zero as $t \to \infty$ along the equilibrium paths in which $v \to \infty$, since they imply $m \to 0$. It is perhaps interesting to note that when $v \to 0$ instead, this equation implies that b eventually grows at a rate approaching β, so that transversality, which we already know is violated on such paths because of the behavior of m, is also a problem from the point of view of the behavior of b.

(c) **Extra credit.** Show that the explosive equilibria make money balances go to zero in finite time.

By using the definition of Z $(**)$ in $(*)$ we can rewrite it entirely in terms of v:

$$\dot{v} = \frac{v(1+2\gamma v)(\gamma v^2 - \beta)}{1+4\gamma v}\,.$$

We could divide both sides of this equation by the right-hand-side, which would give us $\dot{v}$ times a rational polynomial in v on the left. A partial fraction expansion of the coefficient of v would then allow an explicit integration. More indirectly, we can note that the right-hand-side above, for large v, is of the order of v^3. In particular, there will be some positive θ such that θv^3 is less than the right hand side for all v large enough. Therefore we can generate a time path that lies below the v time path by picking a date t_0 at which the right-hand-side exceeds θv^3, then constructing a solution to $\dot{v} = \theta v^3$ with v_{t_0} as initial condition. But $\dot{v} = \theta v^3$ is easily solved, and we see that its positive solutions are of the form $v_t = .5/\sqrt{A-t}$. So the lower bound on v goes to infinity in finite time and v itself must also do so. The fact that on these equilibrium paths money becomes valueless in finite time does not create any problem with their being equilibria. At every moment along these paths everyone is optimizing and everyone understands that money is on its way to becoming worthless. In this particular model utility becomes $-\infty$ at this point, because transactions costs absorb all of the endowment and we are using log utility. But that does not create any market forces that prevent the disaster. Preventing the disaster requires some kind of fiscal backing for the value of money.

(d) Suppose the monetary authority decides, as a surprise to the public, to change the money growth from $-\frac{1}{2}\%$ per year to 3% per year and keep the new higher rate of money growth in place forever. Assume the public immediately understands and believes in the permanence of the new policy. Assume $\gamma = .00018$, which is roughly right for the US, based on the ratio of high-powered money (unadjusted for reserve requirements) to nominal GDP and the current nominal interest rate. Also assume that the discount rate $\beta = .02$. Show that the rise in the inflation rate may require the bank to instantaneously absorb money balances by selling off some of its bond holdings, if it is to implement the policy without an infusion of capital from the treasury and without allowing a jump in the price level. What is the minimum ratio of marketable bonds to money (assuming all its liabilities are money) that it will require to implement the changed policy without a jump in the price level?

I should have said that you should assume the stable equilibrium prevails before and after the change in money growth rate. Velocity in the first regime is $\sqrt{.015/.00018} = 9.13$. In the second regime it is $\sqrt{.05/.00018} = 16.67$.

Money balances don't shrink quite in proportion to the change in velocity, but we can use $C = Y/(1 + \gamma v)$ to find $v + \gamma v^2 = PY/M$. So with Y fixed, to avoid a jump in P, M has to be decreased in proportion to $v + \gamma v^2$. This turns out to be a 45% decrease, and therefore to implement it with open market operations the bank has to have at least 45% of the outstanding high powered money M on hand in marketable assets — or else have fiscal backing.

Note: The Bank of Japan was urged, during Japan's long period of slow growth and deflation, to aggressively purchase long term debt to end the deflation. In our example above, if the central bank's assets were all consols, their value would have fallen by the factor .015/.05=.30, i.e. a %70 decline in asset values. Under rational expectations and commitment, this would happen as soon as the policy were announced. Thus with only infinitely long-term bonds on its balance sheet as assets, the central bank could not initiate a policy of permanent increase in money growth of this magnitude unless it began with assets well in excess of the outstanding money stock. The Bank of Japan did not initially have much long-term debt on the asset side of its balance sheet, but it was thinking about this kind of scenario, plus reluctance to ask for fiscal backing, that was part of the explanation why it was not more aggressive in purchasing long term debt.

(2) **Simple RI problem** Consider the problem of choosing X_t to minimize $E_t[(Y_t - X_t)^2]$. The problem starts at $t = 0$, when the optimizer has as an initial probability distribution $Y_0 \sim N(\hat{Y}_0, \sigma_0^2)$ and must choose X_0 given that distribution. The optimizer knows that $Y_t = Y_{t-1} + \varepsilon_t$, all t, where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ for all t and ε_t is independent of all values of Y_s and ε_s for $s < t$. At $t = 1$ the optimizer can reduce his uncertainty about Y_1 before choosing X_1, subject to a constraint that the mutual information between X_1 and Y_1 conditional on information at $t = 0$ be less than or equal to κ bits. This process continues indefinitely, with the optimizer at each t being able to improve his knowledge of Y_t subject to the same form of mutual information constraint, and with the same κ.

(a) Show that the agent will choose to make the conditional distribution of Y_t given information at t normal at every date.

This is an argument already given in class. At each date we solve

$$\min_f \int (y - x)^2 f(x, y) dx\, dy$$

subject to

$$\int f(x,y)dx = g(y)$$

$$f(x,y) \geq 0$$

$$\int f(x,y)\log(f(x,y))dx\,dy - \int \log(\int f(x',y)dx')f(x,y)dx\,dy - \int g(x)\log(g(x))dx \leq \kappa\,,$$

where g is the marginal pdf of y with which we enter the period, given information up to that time.

Assuming $f(x,y) > 0$ everywhere, the FOC is

$$(y-x)^2 = \mu(x) - \lambda(\log(f(x,y)) + 1 - \log\left(\int f(x',y)dx'\right) - 1) = \lambda\log\left(\frac{f(x,y)}{\int f(x',y)dx'}\right).$$

This implies

$$\frac{f(x,y)}{\int f(x',y)dx'} = \sqrt{\frac{\nu}{2\pi}}e^{-\frac{\nu}{2}(y-x)^2}\,,$$

i.e. that the conditional distribution of $y \mid x$ is normal. This problem has a concave objective function (if thought of as a maximization of $-(y-x)^2$) and convex constraints, so a solution to its FOC's is a solution to the problem. So long as there is some marginal pdf h for X such that $\int h(x)\phi(y-x;\nu^{-2})dx = g(y)$, this is the solution. For $g(y)$ normal, there certainly is always such an h — h itself normal works. And the conditional distribution of $y \mid x$ becomes next period's $g(y)$, so joint normality is preserved.

(b) Derive an equation that shows how σ_t^2, the conditional variance of Y_t given information at t, evolves over time, and find its steady state value as a function of κ and σ_ε^2.

It will always be optimal to reduce conditional variance of Y as much as possible, given κ. The mutual information flow between X_t and Y_t is $\log(\tau_t/\sigma_t)$, where τ_t is the standard deviation of the conditional distribution of Y_t given information up to time t, and σ_t is the conditional standard deviation of Y_t after the information flow at t. We also have $\tau_{t+1}^2 = \sigma_t^2 + \sigma_\varepsilon^2$. Thus $\sigma_{t+1}^2 = 2^{-2\kappa}(\sigma_t^2 + \sigma_\varepsilon^2)$. This difference equation has a steady state at

$$\sigma^2 = \frac{\sigma_\varepsilon^2 2^{-2\kappa}}{1-2^{-2\kappa}}$$

$$\tau^2 = \frac{\sigma_\varepsilon^2}{1-2^{-2\kappa}}\,.$$