TRACKING EXERCISE

1. Overview of answer

People seem not to have had difficulty with the first part of the problem, with $\alpha = 0$, which is a pure tracking problem case. With $\alpha = 1$, however, the problem is no longer a pure tracking problem. In the in-class discussion students argued that setting $x_t = \hat{y}_t$ is still the solution for the control, and proceeded from there. This is, however, not correct.

It turns out (I didn't realize this in setting the problem) that $\alpha = 1$ is a knife-edge special case in which the solution dynamics are in a sense degenerate, so $x_t = \hat{\hat{y}}_t$ is *almost* a correct solution. With other values of α, it is more clear that $x_t = \hat{y}_t$ is not a solution.

That $x_t = y_t$ is not a solution for the certainty-equivalent (CE) problem is clear once we recognize that the state evolution equation (16) can be written as

$$y_t - x_t = \rho y_{t-1} \,.$$

Everything on the right-hand side of this equation at $t = 0$ is given at that date. The choice of x_0 cannot affect y_{-1}. Choosing $x_0 = y_0$ is impossible unless y_{-1} happens to be zero. As we will show below, the optimal decision rule for the CE problem is $x_t = -y_t$. This has no effect on time-0 losses $\|y_0 - x_0\|^2$, because that is fixed at $\rho^2 y_{-1}^2$. But it does mean that $y_0 = 0$ and thus that, with the optimal decision rule, $y_1 = x_1 = 0$, and indeed that thereafter $y_t = x_t = 0$ for all $t > 0$. This also means that for $t > 0$, $y_t = x_t$, so in a sense the solution does set $y_t = x_t$, but not in the initial $t = 0$ period. The decision rule that does hold at every date is $y_t = -x_t$.

The problem is tricky because I meant to put x_{t-1}, not x_t, on the right-hand-side of (16). In that case y would be the state variable, y_0 not $y_0 - x_0$, would be given at time 0, and we would be right to expect a solution that made x_t a linear function of y_t, the state. To get the problem as stated into a standard form with lagged control, not current control, on the right-hand side of the state evolution equation, we must change notation, letting $z_t = y_t - \alpha x_t$ be the state. Then the system becomes

$$\max_{\{z_t, \sigma_t^2, x_t\}} E_{t-1}\Big[\sum_{t=0}^{\infty} \beta^t \|z_t - (1-\alpha)x_t\|^2\Big] - \theta(\log W_t - \log \sigma_t^2) \tag{1}$$

$$\text{subject to} \quad z_t = \rho z_{t-1} + \rho\alpha x_{t-1} + \varepsilon_t \tag{2}$$

$$W_t = \operatorname{Var}(z_t \mid \mathcal{I}_{t-1}) = \rho^2 \sigma_{t-1}^2 + 1 \tag{3}$$

$$W_t - \sigma_t^2 \geq 0 \,, \tag{4}$$

Date: November 15, 2020.

2. Certainty equivalent problem

The FOC's for a Lagrangian approach to the CE version of this problem are

$$\partial x : \quad -2(z_t - (1-\alpha)x_t)(1-\alpha) = -\beta\rho\alpha\lambda_{t+1} \tag{5}$$

$$\partial z : \quad 2(z_t - (1-\alpha)x_t) = \lambda_t - \beta\rho\lambda_{t+1}\,. \tag{6}$$

When $\alpha = \rho = 0$, so we are in the pure tracking case of the first question, it is easy to check that the solution is $z_t = y_t = x_t$. More generally, the FOC's reduce to

$$(1-\alpha)(z_t - (1-\alpha)x_t)) = \beta\rho(z_{t+1} - (1-\alpha)x_{t+1})\,. \tag{7}$$

Using the fact that we know the solution will have the form $x_t = \phi z_t$ for some ϕ, this becomes

$$(1-\alpha)z_t = \beta\rho z_{t+1}\,, \tag{8}$$

and the CE state evolution equation becomes

$$z_t = \rho(1+\phi\alpha)z_{t-1}\,. \tag{9}$$

These two equations then imply

$$\phi = \left(\frac{1-\alpha}{\beta\rho} - 1\right)\frac{1}{\alpha}\,. \tag{10}$$

When $\alpha = 1$, this makes $\phi = -1$, as claimed above.

3. Problem with CE solution plugged in for x_t

Using the $x_t = \phi\hat{z}_t$ CE solution, we can rewrite the objective function (1) as

$$E_{-1}\left[\sum_{t=0}^{\infty}\beta^t\left((z_t - \hat{z}_t)^2 + (1-\phi+\alpha\phi)^2\hat{z}^2 + \theta(\log W_t - \log\sigma_t^2)\right)\right]$$
$$= E_{-1}\left[\sum_{t=0}^{\infty}\beta^t\left(\sigma_t^2 + (1-\phi+\alpha\phi)^2\,\mathrm{Var}_{-1}(\hat{z}_t) + \theta(\log W_t - \log\sigma_t^2)\right)\right]\,. \tag{11}$$

In the pure tracking case, where $\alpha = 0$ and $\phi = 1$, the $\mathrm{Var}_{-1}\,\hat{z}_t$ drops out, and we end up with a deterministic problem in σ_t^2 and W_t. But when $\alpha \neq 0$, even when it is 1, the $\mathrm{Var}_{-1}\,\hat{z}_t$ is present. It does depend on W_t and z_t, but it depends on the whole sequence of them at dates between 0 and t. There is therefore much work remaining to get the objective function into a standard form, depending only on the W_t, σ_t^2 sequence.

Here again, $\alpha = 1$ is a knife-edge special case. We have verified that in this case $E_{t-1}\hat{z}_t = 0$, so $\mathrm{Var}_{-1}(z_t) = \mathrm{Var}_{t-1}(\hat{z}_t) = W_t - \sigma_t^2$. Thus in this case the objective function becomes

$$E_{-1}\left[\sum_{t=0}^{\infty}\beta^t W_t\right] \tag{12}$$

and allowing straightforward (but tedious! I'm only 80% sure it's correct) solution, when combined with (3), to obtain

$$\sigma_t^2 = \frac{W_t}{W_t + \theta}\frac{\theta}{\beta\rho^2} \tag{13}$$

$$\therefore \sigma_t^2 = \frac{(\rho^2\sigma_{t-1}^2 + 1)\theta}{(\rho^2\sigma_{t-1}^2 + 1 + \theta)\beta\rho^2)}\,. \tag{14}$$

So we see that in general σ_t^2 varies over time and converges to a steady state. This contrasts with the $\alpha = 0$ case, where the solution jumps to the steady state at $t = 1$. So long as $\beta\rho^2 < 1$, there will be a θ so large that the no-forgetting constraint always binds.

4. Original problem statement

The problem is

$$\max_{\{y_t\}.\{\sigma_t^2\}} E_{t-1}\Big[\sum_{t=0}^{\infty}\beta^t \left\|y_t - x_t\right\|^2\Big] - \theta(\log W_t - \log\sigma_t^2) \tag{15}$$

$$\text{subject to} \quad y_t = \rho y_{t-1} + \alpha x_t + \varepsilon_t \tag{16}$$

$$W_t = \mathrm{Var}(y_t \mid \mathcal{I}_{t-1}) = \rho^2\sigma_{t-1}^2 + 1 \tag{17}$$

$$W_t - \sigma_t^2 \geq 0\,, \tag{18}$$

where $\varepsilon_t \mid \{y_s, x_s, s \geq 1\} \sim N(0,1)$.

x_t is chosen with the $\mathcal{I}_t$ information set and σ_t^2 is chosen with the $\mathcal{I}_{t-1}$ information set. The information set $\mathcal{I}_t$ consists of al values of x_s dated $s = t$ or earlier, or equivalently the value of all the noisy signals about y received up to time t.

(1) Solve the problem for $\beta = .9$, $\rho = .9$, $\alpha = 0$ and various values of θ. Is there a value of θ so high that no information is collected? So low that the problem is deterministic?

(2) Write out the Bellman equation for this problem for the same parameter values except now with $\alpha = 1$. Explain why your previous solution method doesn't work on this version of the problem.