LINEAR QUADRATIC RATIONAL INATTENTION EXERCISE

Consider a monopolist facing the linear demand curve

$$Q = a - .5P \tag{1}$$

and with unit costs c. We suppose that the monopolist must set the price but is uncertain about the level of demand a and the unit cost c. Before processing any information, the monopolist believes

$$\begin{bmatrix} a \\ c \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} .25 & 0 \\ 0 & .25 \end{bmatrix} \right) . \tag{2}$$

With the cost of information processing given by θ times mutual information between the decision variable P and the uncertain information (a, c), the monopolist's objective function is

$$E[(a - .5P)(P - c)] - \theta\big(\log(|\Sigma|) - \log(|\Omega|)\big) , \tag{3}$$

where Σ is the initial covariance matrix of (a, c) as specified in (2) and Ω is the conditional covariance matrix of (a, c) given information collected.

Determine how Ω varies with θ.

Hint: If you map this into the general notation of the notes and look for the solution when the information constraint does not bind, you will find that it seems to be the inverse of a singular matrix. That is because in this problem the monopolist optimally responds to only a single linear combination of a and c and therefore collects information on (reduces variance of) only that linear combination. Without the no-forgetting constraint, the monopolist would "forget" (increase the variance of) the component of (a, c) that is unimportant to allow further reduction in the variance of the component that is important.

So apply certainty equivalence to see what linear combination of (a, c) matters, determine how the variance of that varies with θ, and characterize the behavior of Ω that emerges.

Note also that, because this setup is linear-quadratic and Gaussian, profits, costs, output and prices can all with some, hopefully small, probability be negative. Do not try to enforce positivity constraints on them.

Answer

In the problem without information constraints the monopolist wants to maximize

$$-.5P^2 + (a + .5c)P - ca ,$$

which from the FOC's give us $P = a + .5c$. The certainty-equivalence principal tells us that in the LQ problem with c and a uncertain, the monopolist will set $P = E[a] + .5E[c]$. The value of the objective function does depend on both a and c separately, not just on $a + .5c$, but the only part of the objective function the price-setter can affect depends only on $a + .5c$.

Date: October 4, 2017.

Let

$$W = \begin{bmatrix} 1 & .5 \\ -.5 & 1 \end{bmatrix}$$

and let

$$x = W \begin{bmatrix} a \\ c \end{bmatrix} .$$

then

$$\mathrm{Var}(x) = \begin{bmatrix} \frac{5}{16} & 0 \\ 0 & \frac{5}{16} \end{bmatrix}$$

and x_1, the first component of the x vector, is the linear combination of a and c we are interested in. If we collect information only on x_1, we reduce its variance while leaving the variance of x_2 unchanged. If the post-information-collection variance of x_1 is σ^2, the cost of information is

$$\theta \cdot (\log(5/4) - \log(\sigma^2) .$$

(Note that the true information cost in nits is one half of what is specified in the problem, but this is just a units of measurement issue. Also note that comparing variances or covariance matrices to arrive at information costs gives an answer that is invariant to linear transformations of the random vectors considered.)

Expected profits with optimally chosen P are $.5x_1^2 - \sigma^2/2$. Taking profits minus information costs and applying first order conditions tells us that the optimal choice of σ^2 is 2θ. But no-forgetting tells us that $\sigma^2 < 5/16$, that is, the variance must go down with information collection. But this in turn implies that it will be optimal to collect no information whenever $\theta > 5/32$. As information costs shrink, the uncertainty about $a + .5c$ shrinks toward zero, and in the limit of free information, $\mathrm{Var}(a, c)$ is

$$\begin{bmatrix} .05 & -.1 \\ -.1 & .2 \end{bmatrix} ,$$

which is, of course, singular. This also illustrates the point that, despite the initial independence of uncertainty about a and c, the price-setter chooses information that makes beliefs about a and c negatively correlated conditional on the information.