Simple FTPL, continuous time

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- It exercises, and slightly generalizes, the deterministic continuous time optimization tools you learned from Ben Moll.
- Analysis of uniqueness and existence issues is a little cleaner in continuous time.
- Doesn't hurt to review.

The slightly generalized tools

- We avoid having to locate "state" variables, which in complex models is not always easy.
- We avoid having to create difficult-to-interpret state variables.
- We gain some insight into which are "jump" variables.

A general problem

$$\max_{x} \int_0^\infty e^{-\beta t} U(x, \dot{x}, z)\, dt \quad \text{subject to}$$

$$g(x, \dot{x}, z) \leq 0\,.$$

FOC's

$$H(x,\dot{x},z,\lambda) = e^{-\beta t}\big(U(x,\dot{x},z) - \lambda g(x,\dot{x},z)\big)$$

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial \dot{x}} = 0$$

$$\text{TVC: } \limsup_{t\to\infty}\left(e^{-\beta t}\frac{\partial H}{\partial \dot{x}}\cdot(x - x')\right) \le 0\,, \quad \text{for all feasible } x'$$

$$g(x,\dot{x},z)\lambda = 0\,, \quad \lambda \ge 0\,.$$

These are, under some regularity conditions, necessary and sufficient conditions for an optimum in problems where U is concave and g is convex. Note that this depends on the constraints being inequalities, so that λ is always positive.

Continuous time flex-price FTPL model

Household:

$$\max_{C,B,M} \int_0^\infty e^{-\beta t} \log C_t \, dt \quad \text{subject to}$$

$$C \cdot \left(1 + \gamma \frac{PC}{M}\right) + \frac{\dot{B} + \dot{M}}{P} \leq Y + \frac{rB}{P} - \tau$$

$$B \geq 0 \quad M \geq 0 \, .$$

Note: The constraint is jointly concave in the choice variables C, B and M, in the sense that if it is satisfied for two C, B, M paths, it is also satisfied (as an inequality) for convex linear combinations of the two paths. Can you show this?

More dotted variables than constraints

The household budget constraint contains both $\dot{B}$ and $\dot{M}$. To get it into a form with a single "state" variable on the left and exogenous variables and "controls" on the right, we would need to define $W = B + M$ and replace rB/P on the right by $r(W - M)/P$. This would make it clear that what is fixed at time zero is the value of $B + M$ inherited from the past. The household cannot make wealth discontinuously increase at time zero. It can, however, make M or B change discontinuously at time zero, so long as $B + M$ does not change. We maintain this kind of interpretation in the government budget constraint below and in the Taylor rule monetary policy equation.

The FOC's, on the other hand, though they include time derivatives, do not imply that these time derivatives force continuity at time zero of the linear combinations of variables appearing as time derivatives. They are "forward looking" equations, in the sense that derivatives appearing in them are right-derivatives.

Household FOC's

For an interior solution, using the notation $v = PC/M$ for velocity,

$$\partial C: \ \ \frac{1}{C} = \lambda(1 + 2\gamma v)$$

$$\partial B: \ \ \left(\beta + \frac{\dot{P}}{P} - \frac{\dot{\lambda}}{\lambda}\right)\frac{\lambda}{P} = \frac{r\lambda}{P}$$

$$\partial M: \ \ \frac{\lambda}{P}\left(\beta + \frac{\dot{P}}{P} - \frac{\dot{\lambda}}{\lambda}\right) = \gamma v^2 \frac{\lambda}{P}$$

$$\text{TVC}: \ \ \liminf e^{-\beta t} - \frac{\lambda}{P}(B_t - B_t' + M_t - M_t') \leq 0\,, \ \text{all feasible } B', M'$$

Reorganizing household FOC's

$$\text{intertemporal :} \qquad r - \frac{\dot{P}}{P} - \beta = \frac{\dot{C}}{C} + \frac{2\gamma\dot{v}}{1 + 2\gamma v}$$

$$\text{liquidity preference :} \qquad r = \gamma v^2$$

Government

$$
\begin{aligned}
\text{GBC :} \quad & \dot{B} + \dot{M} + \tau P = rB \\
\text{Fiscal policy :} \quad & \begin{cases} \text{active :} & \tau \equiv \bar{\tau} \\ \text{passive :} & \tau = -\phi_0 + \phi_1 \frac{B}{P} \end{cases} \\
\text{Monetary policy :} \quad & \begin{cases} \text{active} & \dot{r} = \theta_0 \left(\theta_1 \frac{\dot{P}}{P} - r + \beta \right) \text{ or } M \equiv \bar{M} \\ \text{passive} & r \equiv \bar{r}\,. \end{cases}
\end{aligned}
$$

What the Taylor rule allows to jump

Maintaining the same interpretation of a constraint containing two derivatives in linear combination that we applied to the household budget constraint, we see that the Taylor rule implies that r and $\log P$ can both jump at time 0, but $r - \theta_0\theta_1 \log P$ cannot. That is, policy implies that if $\log P$ jumps by Δp at time 0, r jumps by $\Delta r = \theta_0\theta_1\Delta p$. This follows naturally from thinking about what the Taylor rule implies would happen to r if inflation where finite, but very high, over a very short period of time.

Active Fiscal, Passive Money

The policies we have given these labels (they are special cases of more general categories) make it particularly easy to show that, so long as $\bar{\tau}$ is positive, there is an equlibrium with a uniquely determined initial price level. From here on we assume Y is constant.

The fixed r policy, because of the liquidity preference relation, implies $v \equiv \bar{v} = \sqrt{r/\gamma}$.

Subtracting the GBC from the household constraint gives us the social resource constraint (SRC) $C(1 + \gamma v) = Y$, and then, since Y is constant, C is constant.

Constant C and v imply, via the intertemporal FOC, that $\bar{r} = \beta + \dot{P}/P$, so inflation is constant and real balances $\bar{m} = M/P = C/\bar{v}$ are constant.

Constancy of C and v imply, via the definition of v, that $\dot{M}/M = \dot{P}/P = r - \beta$.

Using the GBC

Rewrite the GBC in terms of $b = B/P$:

$$\dot{b} + b\frac{\dot{P}}{P} + \frac{\dot{M}}{P} = rb - \tau \; .$$

Using what we already know about the growth rate of M and P and the constancy of v, this becomes

$$\dot{b} = \beta b + (\bar{r} - \beta)\bar{m} - \tau \; .$$

Unstable GBC

$$\dot{b} = \beta b + (\bar{r} - \beta)\bar{m} - \tau \; .$$

This is an unstable differential equation in b. An upward explosive solution grows fast enough to violate household transversality. A downward explosive solution eventually becomes negative, violating the constraint that people can't borrow from the government.

Unique non-explosive solution

$$b \equiv \bar{b} = \frac{\tau + (\bar{r} - \beta)\bar{m}}{\beta}$$

If $\bar{r} = \beta$, so the equilibrium has no inflation, this becomes just

$$b = \bar{\tau}/\beta \; .$$

The formula can be read as stating that the real debt is the discounted present value of primary surpluses (taxes net of spending other than interest) plus revenue from seignorage ($\dot{M}/P$).

Unique initial price level

Recall that $B + M$ cannot jump at time zero. Let $W_0 = B_0 + M_0$. Then

$$\frac{W_0}{P_0} = \bar{b} + \bar{m}$$

We have already found values for everything on the right side of this equation, and W_0 is given by history. Thus there is only one initial price level P_0 consistent with equilibrium.