FTPL EXERCISE

Consider a model (all the models in this exercise are close to what we discussed in class) in which the agent's problem is

$$\max_{C,B,M} \int_0^\infty e^{-\beta t} \log C_t \, dt \quad \text{subject to} \tag{1}$$

$$C \cdot \left(1 + \frac{\gamma v}{1+v}\right) + \frac{\dot{B} + \dot{M}}{P} + \tau \leq \frac{rB}{P} + Y \tag{2}$$

$$B \geq 0, \quad M \geq 0\,. \tag{3}$$

Note that as usual we are using the notation $v = PC/M$, and we assume that after possible surprises at $t = 0$, people have perfect foresight about the future paths of P, r, and τ.

The government budget constraint is

$$\frac{\dot{B} + \dot{M}}{P} = \frac{rB}{P} - \tau\,. \tag{4}$$

(1) Determine whether the constraint set here is convex in the time paths of C, B and M, and explain how you know.

One way to do this is to check that the second derivative matrix of the constraint with respect to C, M and B is positive definite. This guarantees that the constraint function is convex, and thus that the constraint set is convex. This only works here because $\dot{C}$ does not enter the constraint and $\dot{B}$ and $\dot{M}$ enter linearly. B also enters linearly, so we need only check the second derivative matrix with respect to C and M, which is

$$\frac{2\gamma v}{(1+v)^3} \begin{bmatrix} \frac{1}{C} & -\frac{1}{M} \\ -\frac{1}{M} & \frac{C}{M^2} \end{bmatrix}.$$

It is easy to see that this is singular and that both diagonal elements are positive. It is therefore positive semi-definite and implies convexity.

(2) Show that if the policy rules are $r \equiv \bar{r}$, $\tau \equiv \bar{\tau}$, this model has a unique initial price level. Show how initial price level, inflation rate time path, and interest rate path depend on the policy parameters.

The FOC's give us

$$\lambda = \frac{1}{C(1 + \gamma - \frac{\gamma}{(1+v)^2})} \tag{5}$$

$$r = \frac{\gamma v^2}{(1+v)^2} \tag{6}$$

$$-\frac{\dot{\lambda}}{\lambda} = r - \frac{\dot{P}}{P} - \beta\,. \tag{7}$$

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The second of these tells us that with r pegged, v is pegged. The SRC tells us that this means C is pegged. From the first of the listed FOC's this means λ is constant, and thus from the third FOC that the inflation rate is pegged at $r - \beta$. The real form of the the government budget constraint is

$$\dot{b} = \left(r - \frac{\dot{P}}{P}\right) b - \frac{\dot{M}}{M}\frac{M}{P} - \tau\,.$$

Constant v and C implies that $\dot{M}/M$ matches the constant $\dot{P}/P$, and $M/P = C/v$ is also constant. Call the $\dot{M}/P$ term s, for seigniorage. Then the government budget constraint tells us that, since the real rate $r - \dot{P}/P$ is constant at β, the real debt b must always be $(\tau + s)/\beta$ if it is not to explode upward at the rate β (violating transversality) or downward at that rate (violating feasibility). But $B + M$ cannot jump at time zero and $M = PC/v$. Thus we have

$$B + M = (b + C/v) \cdot P\,.$$

The left-hand side is uniquely determined by history at the initial date, and everything on the right-hand side except P is pinned down by equilibrium conditions. So there is only one initial P consistent with equilibrium. This does require that $\tau + s$ be positive. Note that the second FOC limits the range of r values that can be feasibly pegged. $r < \gamma$ is required. Because transactions costs are bounded, a high enough nominal interest rate will make people stop using money.

(3) Suppose instead the policy rules are $M \equiv \bar{M}$ and $\tau = -\phi_0 + \phi_1 b$, where $b = B/P$ is real debt. Assume $\phi_0 > 0$, $\phi_1 > \beta$. Is there an equilibrium with constant v? Is the initial price level uniquely determined? Answers to these questions may well vary depending on the values of β and γ.

From the FOC's, and using the fact that $\dot{C}/C + \dot{P}/P = \dot{v}/v$ with $M \equiv \bar{M}$, we arrive at

$$-\frac{\dot{\lambda}}{\lambda} = \frac{\dot{v}}{v} + \frac{2\gamma\dot{v}}{(1+v)^3} = \frac{\gamma v^2}{(1+v)^2} - \beta = r - \beta\,. \qquad (*)$$

For there to be a solution with $\dot{v} = 0$, $v \equiv \bar{v}$ would have to solve

$$\frac{\gamma\bar{v}^2}{(1+\bar{v})^2} = \beta\,.$$

This is possible only if $\gamma > \beta$. If this condition is satisfied, the equilibrium with constant v implies $r = \beta$, and thus that inflation is zero, with the price level determined by

$$\frac{PC}{\bar{M}} = \frac{PY}{(1 + \gamma v/(1+v))\bar{M}} = \bar{v}\,.$$

Transversality is clearly satisfied, since real balances will be constant and real debt is guaranteed to converge to $\phi_0/(\beta - \phi_1)$.

What if $v_0 < \bar{v}$ or $\gamma < \beta$? Then $\dot{v}$ is always negative, according to $(*)$. Furthermore, that equation can be rearranged and integrated with respect to v as we did in class to conclude that v goes to zero as $t \to \infty$, but does not reach zero in finite time because the integral is $O(v^{-1})$ as $v \to 0$. In the limit, as $v \to 0$, $(*)$ implies that $\dot{v}/v \to -\beta$, which in turn implies, because of constant M and stable C, that real balances grow at a rate approaching β. This rules out these paths as equilibria, because the wealth growing at the real interest rate would violate transversality. Agents seeing the initial low velocity would feel too wealthy to be consistent with equilibrium; they would try to spend down their balances, which would either bring velocity back up to $\bar{v}$ or above, or (always when $\gamma < \beta$) lead to valueless money.

If $v_0 > \bar{v}$, rearranging $(*)$ and integrating over v tells that, unlike the case of the linear γv transactions cost model discussed in class, here v grows forever, without reaching infinity in finite time, because the integral over v is $O(v^{-1})$ as $v \to \infty$. As $v \to \infty$ $\dot{v}/v$ converges to $\gamma - \beta$, which implies that eventually inflation converges to this rate. Real balances shrink exponentially toward zero and real debt is again kept stable by the fiscal rule, so the TVC is satisfied. We conclude that any initial $v_0 > \bar{v}$ corresponds to an equilibrium, in which real balances shrink exponentially toward zero.

(4) Consider a version of the model in which $\gamma = 0$, i.e. there is no money. Suppose monetary policy is a Taylor rule and fiscal policy pegs the primary surplus (a version of active money, active fiscal). That is, specifically,

$$\dot{r} = \theta_0 \left(\theta_1 \frac{\dot{P}}{P} - r - \beta \right) , \quad \theta_0 > 0 , \quad \theta_1 > 1 \tag{8}$$

$$\tau \equiv \bar{\tau} . \tag{9}$$

Does an equilibrium exist? If so, is it unique and how does the economy behave in it (or them)?

The Taylor rule (8) has a typo. I meant it to have "$+\beta$" where it now reads "$-\beta$". The typo makes the answer messier, without changing the basic results — the typo implies a constant, non-zero inflation rate in the non-explosive solution. In what follows I assume the typo corrected.

The SRC, since there are no transactions costs, implies constant $C = Y$. This in turn implies from the FOC's that

$$r - \frac{\dot{P}}{P} = \beta .$$

Using this result in the Taylor rule and writing $\pi = \dot{P}/P$, we get

$$\dot{\pi} = \theta_0 (\theta_1 - 1) \pi .$$

This is an unstable equation with steady state $\pi = 0$. As we have discussed in class, if the Taylor rule is assumed to hold continuously, it allows jumps in r and P at time zero, but only if the jumps satisfy $\Delta r = \theta_0 \theta_1 \Delta P$. This means there is a unique initial P_0 consistent with the stable path for π. If, before $t = 0$, $r \neq \beta$, then r must jump to β and P must jump from its pre-0 value by $\Delta P = (\beta - r)/(\theta_0 \theta_1)$ – if we are to be in the equilibrium with non-explosive inflation.

But when we consider the government budget constraint, we see that we can calculate a unique P_0 consistent with stable real debt from

$$\frac{B}{P} = \frac{\bar{\tau}}{\beta} .$$

Since B cannot jump at time 0, P_0 is uniquely determined.

It would be an unlikely accident if the P calculated this way from the GBC matched that consistent with stable inflation. The stability of b is necessary for satisfaction of the TVC, so any equilibrium must have P_0 matching the value required for stability of b. If this is different from the P_0 consistent with stable inflation, the inflation rate will grow exponentially upward or downward. The upward explosions are certainly valid equilibria. All FOC's and theTVC of the private agents are satisfied. The downward explosions imply ever greater negative inflation rates. Nothing we have said explicitly in the model rules this out. There is no currency to provide an arbitrage-induced zero floor on the nominal interest rate. Government debt with a negative interest rate is feasible, at least if implemented as zero-coupon bonds. The bonds simply are redeemed at maturity for fewer dollars than they cost at purchase.

So, the conclusion is that equilibria exist and are not unique. One has stable (zero in the typo-corrected version) inflation, the others have exponentially growing inflation or deflation. Real allocations are the same in all the equilibria. It is worth noting that in the upward explosive equilibria the conventional, nominal government deficits explode upward, while in the downward explosive ones conventional nominal surpluses explode upward. This may be reason to think that it would be more realistic to suppose that τ reacts to inflation, or the nominal deficit, as well as to b, at least at high rates of inflation or deflation. This could eliminate the explosive paths as equilibria.