TAKE-HOME EXAM

The exam is due at 9AM Thursday, January 19, preferably by electronic submission to both sims@princeton.edu and moll@princeton.edu. Paper submissions are allowed, and should go to Jen Bello in JRRB 273.

Unlike on problem sets, collaboration is not allowed on the takehome, and you are not to discuss the exam with anyone until after 9AM Thursday.

I. A TRACTABLE THEORY OF WEALTH DISTRIBUTION IN GENERAL EQUILIBRIUM (90 POINTS)

Consider the following economy which can be seen as a simplified version of an Aiyagari (1994) economy or, alternatively, a variant on a Blanchard (1985) - Yaari (1965) economy with some market incompleteness. If you do things correctly, you will be able to solve for the stationary equilibrium of this economy in closed form. Except for the very last subquestion, we will focus on stationary economies only, i.e. we are not concerned with transition dynamics of the aggregate economy.

There is a continuum of individuals who are heterogeneous in their wealth a. Individuals are born with initial wealth $a_0 = 0$ and face a constant Poisson death rate $p > 0$ (as in the Blanchard-Yaari model). As a result the cross-sectional age distribution is exponential and given by $\pi(s) = \Pr(\text{age} = s) = pe^{-ps}$. Longevity risk is the only risk in the economy. Individuals discount the future at rate $\tilde{\rho} > 0$ and have preferences given by

$$\mathbb{E}_0 \int_0^\infty e^{-\tilde{\rho} s} u(c(s))ds = \int_0^\infty e^{-\rho s} u(c(s))ds, \quad \rho := \tilde{\rho} + p, \tag{1}$$

where we have used that the probability of surviving past age s is $\Pr(\text{age} \geq s) = e^{-ps}$ and where $\rho := \tilde{\rho} + p$ can be interpreted as the "effective discount rate." The period utility function is given by

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0.$$

Individuals inelastically supply one unit of labor. An individual's budget constraint is

$$\dot{a}(s) = w + ra(s) - c(s). \tag{2}$$

Here w is the common wage and r the common interest rate. Since we are concerned with the stationary equilibrium only, w and r are scalars. Individuals face borrowing constraints given by

$$a(s) \geq -\frac{w}{r}. \tag{3}$$

Individuals maximize (1) subject to (2), (3) and with $a(0) = 0$ given. Note that there are "accidental bequests" meaning that, in general, individuals may die with wealth different from zero. For simplicity we here assume that all wealth is lost in the bequest process. This also explains why individuals' wealth at birth is $a(0) = 0$. In contrast to the standard

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Blanchard-Yaari model there are no annuity markets, i.e. longevity risk is not insurable, and hence there is some market incompleteness.

On the production side, there is a representative firm with Cobb-Douglas production function $F(K, L) = K^\alpha L^{1-\alpha}$. Capital K depreciates at rate δ. From firm profit maximization, the stationary wage and interest rate are given by

$$r = F_K(K, L) - \delta, \qquad w = F_L(K, L) \tag{4}$$

In equilibrium factor demands equal factor supplies and hence

$$K = K^s, \quad K^s := \int_0^\infty a(s)\pi(s)ds, \tag{5}$$

$$L = 1. \tag{6}$$

This completes the description of the economy. Answer the following question:

(1) Write the Hamilton-Jacobi-Bellman equation for an individual's problem.
(2) Derive an analytic solution to an individual's consumption policy function $c(a)$. (Hint: one strategy is to use a guess-and-verify strategy on the HJB equation from question 1 with a guess for v that is a function of "effective wealth" $a + w/r$. There are also alternative strategies.)
(3) If you answered 2. above correctly, your answer there will imply that savings of an s-year old $\dot{a}(s)$ as a function of her wealth $a(s)$ are given by

$$\dot{a}(s) = \frac{r - \rho}{\gamma}\left(a(s) + \frac{w}{r}\right) \tag{7}$$

so that individuals accumulate wealth whenever $r > \rho$ and vice versa.
   (a) Savings $\dot{a}(s)$ are not necessarily increasing in the interest rate r. Why not?
   (b) If $r < \rho$, do individuals starting at $a(0) = 0$ hit the borrowing constraint $\underline{a} = -w/r$ at some point of their lives? If so, why? If not, why not?
(4) We next turn to characterizing the distribution of wealth. We first characterize it for fixed prices w and r and later endogenize these. For subquestions 4. to 6. assume that $r > \rho$. Recall that wealth follows (7) and that individuals die at rate p in which case they are replaced by a newborn with wealth $a(0) = 0$. The wealth process is therefore intimately related to the "Steindl model" covered in class and featured in Gabaix et al (2016). Under the assumption $r > \rho$, what is the support of the stationary wealth distribution?
(5) Derive a closed form solution for the wealth distribution. In particular show that the distribution of "effective wealth" $x := a + w/r$ is Pareto, $\Pr(\tilde{x} \geq x) = (x/x_0)^{-\zeta}$, and find an analytic solution for the tail parameter ζ.

   (Hint: there are two alternative strategies for doing this. You can either work with the Kolmogorov Forward equation; or, alternatively, you can follow the steps in Jones (2015) `http://web.stanford.edu/~chadj/piketty.pdf` who shows that "exponential growth that occurs over an exponentially distributed amount of time" results in a Pareto distribution. See in particular the online Appendix `http://web.stanford.edu/~chadj/SimpleParetoJEP.pdf`.)
(6) Given effective wealth x follows a power law, wealth a follows an *asymptotic* power law with the same tail exponent. How does the fatness of the tail depend on

(a) the interest rate r?
(b) the discount rate ρ?
(c) the death rate p?
(d) the wage rate w?

Very briefly comment on the relation to Piketty's "$r - g$ theory."

(7) We next turn to characterizing macroeconomic aggregates. First derive an expression for the aggregate capital supply K^s as a function of w, r and parameters. You can do this either by integrating (7) across the population; or you can derive K^s from the wealth distribution in subquestion 5. If you did things correctly, the aggregate capital supply is homogeneous of degree 1 in the wage rate w.

(8) Plot capital supply K^s as a function of r. What can you say about the range in which the equilibrium r will have to lie for there to be an equilibrium with a strictly positive and finite aggregate capital stock?

(9) Equating capital demand and supply, find an equation characterizing the equilibrium interest rate r in terms of exogenous parameters only. (Hint: you can take advantage of the fact that both capital and demand and capital supply are homogeneous of degree 1 in the wage rate w). Using this equation, what can you say about existence and uniqueness of a stationary equilibrium?

(10) For the special case of zero depreciation $\delta = 0$, find an analytic solution for the equilibrium interest rate r. Also find analytic expressions for the equilibrium capital stock K, the wage rate w and aggregate consumption $C = \int_0^\infty c(s)\pi(s)ds$ in terms of exogenous parameters only.

(11) How do changes in the following parameters affect the equilibrium r, K and C and what is the intuition?
(a) the discount rate ρ?
(b) the death rate p?
(c) risk aversion γ?
(d) the capital share α?

(12) Now, return to characterizing the wealth distribution. In contrast to subquestion 6 which was concerned with wealth inequality in partial equilibrium we now do this *in general equilibrium*. For the special case $\delta = 0$ from subquestion 7, derive an expression for the tail parameter ζ *in general equilibrium*. How does the fatness of the tail depend on the following parameters and what is the intuition?
(a) the discount rate ρ?
(b) the death rate p?
(c) risk aversion γ?
(d) the capital share α?

(13) Finally, it is interesting to consider "distributional effects," i.e. the question: how do macroeconomic aggregates depend on the distribution of wealth across individuals? In our economy, how would a one-time redistribution of wealth that results in a fatter tail (lower ζ) affect the aggregate capital stock? Relate this finding to the existing literature. (Note that this question is concerned with transition dynamics but you can answer it without explicitly solving for these.)

II. Rational inattention (45 points)

Here is a stylized static model of a consumption decision with uncertain wealth.

The optimizing person maximizes expected utility minus information processing costs, where utility is $\log(C)$, with C consumption. Information processing costs are α times the mutual information between consumption and wealth. It is impossible to consume more than the amount of available wealth. (This means no joint distribution of consumption and wealth can put positive probability on $C > W$.) Wealth can take on only three values, $\{1, 2, 3\}$. Before processing any information that is available, the person has a probability distribution over wealth, with its three values given probabilities $p_1 = .2$, $p_2 = .5$, and $p_3 = .3$. Consumption also can take on only one of these same three values, i.e. $C \in \{1, 2, 3\}$.

(1) What is the optimal policy if no information at all is collected, and what is the optimal policy if information is free ($\alpha = 0$)?

(2) Find the minimal value of α such that consumption is never 3. Find the minimal value of α such that consumption is never 2. Doing this may be easiest if you write a program to solve the relevant equations. You can also get nearly full credit by fully describing equations to be solved, but because you will need to allow for corner solutions, it may be about as easy just to write the code.

III. FTPL (45 points)

Consider a simple continuous time perfect foresight (except for initial suprises) FTPL model without money. It is sometimes argued that eliminating currency, so that all government debt is interest-bearing, could make monetary policy more effective by eliminating the zero lower bound. You will examine this claim in this model.

The representative person maximizes

$$\int_0^\infty e^{-\beta t} \log(C_t)\, dt \quad \text{subject to} \tag{8}$$

$$C + \frac{\dot{B}}{P} = \frac{rB}{P} + Y - \tau \tag{9}$$

$$B \geq 0\,. \tag{10}$$

B is government debt, C is consumption, P is the price level, $\beta > 0$ is the rate of time discount, r is the nominal interest rate, $Y > 0$ is endowment income, assumed constant, and τ is the level of lump sum taxes.

Monetary policy sets

$$\dot{r} = \gamma\left(\theta \frac{\dot{P}}{P} - r + \beta\right). \tag{11}$$

We assume $\theta > 0$. Fiscal policy sets

$$\tau = -\phi_0 + \phi_1 \frac{B}{P}\,, \tag{12}$$

where $\phi_0 > 0$ and $\phi_1 > \beta$.

(1) Write down the government budget constraint and the social resource constraint.

(2) Show that this model has an equilibrium in which prices do not explode upward or downward and that there is only one such equilibrium.

(3) Does the model rule out equilibria in which the price level explodes upward? In which it explodes downward? (Be sure to consider whether transversality conditions and the $B \geq 0$ constraint are satisfied on any explosive path you claim is an equilibrium.)

(4) Suppose fiscal policy instead set $\phi_1 = 0$ and $\phi_0 = -1$. Is there then an equilibrium in which prices do not explode? Is there then a unique equilibrium?