FTPL in a simple Keynesian model

A representative consumer maximizes

$$\int_0^\infty e^{-\beta t} \log C_t \, dt \quad \text{subject to}$$

$$C_t + \tau + \frac{B_t}{P_t} = Y_t + \frac{rB_t}{P_t}.$$  

There is no uncertainty after the initial date. The consumer chooses the time paths of $C$ and $B$, taking $r$, $P$, $Y$, and $\tau$ as given.  

Government policy fixes $r$ and $\tau$ at positive, constant values. To keep the algebra simple, we assume $r = \beta$. Since this is a Keynesian model, instead of an endowment process or a production function we introduce a Phillips Curve, here an old-fashioned backward-looking one:

$$\dot{p} = \gamma (y - \bar{y}),$$

where $p$ is the log of the price level and $\bar{y}$ is a normal, or full-employment, level of the log of output. Again to keep the algebra simple, we assume $\bar{y} = 0$.

The government budget constraint is 

$$\dot{B} = rB - P\tau.$$  

Note that the Phillips curve is not forward-looking, so it implies that $p$ (and thus also $P$) cannot jump discontinuously at the initial date. Also, the government budget constraint implies that $B$ cannot jump at the initial date.

(a) Display the social resource constraint.

Dividing the government budget constraint by $P$ and subtracting it from the private budget constraint delivers $C_t = Y_t$.

(b) Using the private agent’s optimality conditions and the other equations of this model, derive a differential equation system in real debt $b$ and the log of consumption $c$ that must be satisfied in equilibrium.

The FOC’s are

$$\partial C : \quad \frac{1}{C} = \lambda,$$

$$\dot{\lambda} + \frac{\lambda}{p} \dot{p} + \beta \frac{\lambda}{p} = r \frac{\lambda}{p}.$$  

Solving to eliminate $\lambda$ gives us

$$\frac{\dot{C}}{C} = r - \beta - \frac{\dot{p}}{p}.$$
Using the social resource constraint and the Phillips curve, and using lower case \( c \) for the log of consumption, this gives us

\[
\dot{c} = r - \beta - \gamma (c - \bar{y}). \tag{*}
\]

This equation is forward-looking, as it is based on the B FOC, so it does not rule out initial jumps in \( c \). But it determines the time path of \( c \) from any given initial condition.

The government budget constraint becomes

\[
\dot{b} + b \dot{p} = rb - \tau = \dot{b} + b \gamma (c - \bar{y}).
\]

(c) Linearizing if necessary, determine whether the model has any stable solution and if so, whether it is unique.

The model has a steady state, where \( r = \beta, \dot{c} = 0 \), and hence \( c = \bar{y} \) and \( \dot{p} = 0 \). The private FOC equation \((*)\) is already linear. A steady state for \( b \) therefore requires

\[
b = \bar{b} = \tau / r.
\]

Then linearizing the GBC delivers

\[
\dot{b} + b \gamma (c - \bar{y}) = rb - \tau,
\]

The differential equation \((*)\) above is stable and implies (with \( r = \beta \))

\[
c_t = \bar{y} + (c_0 - \bar{y}) e^{-\gamma t}.
\]

Solving the GBC forward gives us

\[
b_0 = \int_0^\infty e^{-rt} (\tau + \bar{b} \gamma (c - \bar{y})) \, dt = \frac{\tau}{r} + \frac{\bar{b} \gamma (c_0 - \bar{y})}{\gamma + r}. \tag{†}
\]

Since both \( B_0 \) and \( P_0 \) can’t jump, \( b_0 \) is predetermined, so \( c_0 \) must adjust to make the equation above hold. Initial \( \dot{p} \) is then determined by the Phillips curve and \( \dot{c} \) by \((*)\). All initial values are thus uniquely determined.

(d) Determine how initial \( c \) and \( \dot{p} \) move if there is a one-time, unanticipated increase in \( \tau \), with the economy initially in steady state.

Equation \((†)\) involves only future \( \tau \) values, so it holds with the new higher value of \( \tau \) in place. But then it is easy to see that, since \( b_0 \) is fixed, \( c_0 \) must decline to offset the increase in \( \tau \). The fall in \( c_0 \) lowers the inflation rate, thereby increasing the real rate of interest, thereby discounting the larger stream of future \( \tau \)’s more heavily to make their discounted value still match \( b_0 \). It is also possible to derive this conclusion without linearizing the government budget constraint, instead solving it forward in its nonlinear form and differentiating with respect to \( c \) and \( \tau \) at the \( c = \bar{y} \) steady state.