BUBBLE MONEY IN CONTINUOUS TIME EXERCISE

A complete answer to this problem would have been really hard. Here I discuss the model, not in the order of the questions about it on the problem sheet, but starting with the cleaner results and ending with more speculative ones.

I. Restatement of the problem

There is a continuum of people, aged $a \in [0, \infty)$, at every time t. Each individual dies with Poisson probability δ per unit of time, and new age-0 people are born to exactly replace those dying, so population remains constant. Individuals solve

$$\max_{C,M,S} \int_0^\infty e^{-(\beta+\delta)a} \log(C_a)\, da \tag{1}$$

$$\text{subject to:} \quad C_a + \tau_a + \dot{S}_a + \frac{\dot{M}_a}{P_t} = Y_a + \gamma S_a \tag{2}$$

$$M_a \geq 0\,, \quad S_a \geq 0\,, \quad M_0 = \bar{M}_t\,, \tag{3}$$

where the t subscript refers to absolute time and the a subscript to age. The bar over M in $\bar{M}$ means the weighted average of M_a across ages, i.e.

$$\bar{M}_t = \delta \int_0^\infty e^{-\delta a} M_{at}\, da\,. \tag{4}$$

We will often leave the t subscript implicit in M_{at} in what follows. The government sets the τ_a schedule. (In the original problem statement, it was τ_a constant, but this is a little awkward, it turns out, so we discuss a case with τ varying with a below.) We focus on the case $\gamma < 0$, where we expect there may be a role for improving welfare by introducing money and thereby eliminating inefficient storage.

II. Positive taxes imply storage disappears, eventually

The first and simplest result is that, if there is any level of positive taxation used (as implied by the government budget constraint) entirely to retire parts of the money stock, there is no equilibrium in which storage, paying a negative return γ, persists forever. In such an equilibrium, if money is held at all, it must pay the same real return γ as storage. but the government budget constraint is

$$\frac{\dot{\bar{M}}}{P} = \dot{\bar{m}} - \rho\bar{m} = -\bar{\tau}\,, \tag{5}$$

where the bars represent averaging across the population, $\rho = -\dot{P}/P$ is the real return on money balances, and m is real balances. (The original problem statement used τ inconsistently, sometimes to represent a transfer, sometimes to represent a tax. Here positive τ is a tax.) We are maintaining the assumption that the wealth of dying people is transferred directly to newborns, so that they are endowed at birth with the average money-holding across the population. This is a stable differential equation when $\rho < 0$, and it converges to $\bar{m} = \bar{\tau}/\rho$ which, with $\rho < 0$, is negative, which is impossible. If the economy started along such a path, people would realize that they could not sustain their planned consumption path without borrowing from the government, which they cannot do, so they would try to save more, thereby increasing $\bar{m}$. With $\bar{m}$ higher, people would perceive themselves richer, hence consume more, thereby reducing real saving in the form of storage.

III. With any level of positive taxation, there is an equilibrium with valued money

A second definite result is that there is an equilibrium, under some parameter values, in which taxation is positive, there is no storage, and $\bar{m}$ is constant, and for a given level of taxation there is only one such equilibrium. It is a associated with a unique initial price level. The problem statement had $\bar{\tau} = \tau_a$ constant, so people of all ages pay the same tax. Such a policy is feasible only under a restricted range of parameter values. Individual resources and consumption levels decline with age in this economy, while the tax burden under this policy remains fixed. This can easily make the demand for saving so strong that desired saving in the form of money always outstrips supply from dissavers. So we will first consider a modification of the problem in which older people are taxed less. In particular, we will assume that those of age a pay a tax $\tau_a = \tau_0 e^{-\theta a}$. This makes the tax decline at the same rate as people's endowment income.

The government budget constraint tells us that if $\bar{m}$ is constant, the real return $\rho = -\dot{P}/P$ on money must also be constant, and furthermore that $\bar{m} \equiv \bar{\tau}/\rho$. Note that because we are averaging across ages, weighting by population,

$$\bar{\tau} = \int_0^\infty \delta e^{-\delta a} \tau_0 e^{-\theta a} = \frac{\delta \tau_0}{\theta + \delta} \,. \tag{6}$$

The FOC's of the private agent's problem imply that

$$\frac{\dot{C}}{C} = \rho - \beta - \delta$$
$$\therefore C_a = C_0 e^{-(\rho - \beta + \delta) a} \,.$$

At time 0, then, we can solve the private budget constraint forward to obtain

$$m_0 = \bar{m} = \frac{C_0}{\beta + \delta} + \frac{\tau_0}{\theta + \rho} - \frac{Y_0}{\theta + r} \,. \tag{7}$$

This simply states that initial wealth covers the gap in present value between the future consumption and tax streams and the future endowment stream.

At each age, surviving members of the population face a problem of the same form: an initial level of taxation τ_a and real balances m_a and an exponentially declining endowment stream with initial level Y_a. So the same argument that gave us (7) also gives us

$$m_a = \frac{C_a}{\beta+\delta} + \frac{\tau_a}{\theta+\rho} - \frac{Y_a}{\theta+\rho}\,. \tag{8}$$

We can average this equation across ages, weighting by population, to obtain

$$\bar{m} = \frac{\bar{C}}{\beta+\delta} + \frac{\bar{\tau}}{\theta+\rho} - \frac{\bar{Y}}{\theta+\rho}\,. \tag{9}$$

But in equilibrium, since there is no storage, $\bar{C} = \bar{Y}$. And we know that $\bar{m} \equiv \bar{\tau}/\rho$. This lets us derive from (9)

$$\bar{\tau}\left(\frac{1}{\rho} - \frac{1}{\theta+\rho}\right) = \bar{Y}\left(\frac{1}{\beta+\delta} - \frac{1}{\theta+\rho}\right) \tag{10}$$

The left-hand side of this expression is monotone decreasing in ρ, from infinity to zero as ρ goes from zero to infinity. The right-hand side is monotone increasing in ρ and positive for large enough ρ. This means there is a unique positive equilibrium value of ρ. Then we can determine $\bar{m} = \bar{\tau}/\rho$ and from that and (7), determine C_0. Because initial $\bar{M}$ is inherited from the past, having determined $\bar{m}$ we have also determined P_0.

So we have almost shown the result that obtains in the Samuelson two-period-life OG model, i.e. that with any positive level of tax there is a unique initial price level. However there remains a gap in the reasoning. I have assumed in calculating the equilibria that $\bar{m}$ remains constant. I suspect there are no other equilibria, but have not been able to prove that there could not be equilibria with time-varying $\bar{m}$ and (therefore) time-varying ρ.

IV. No-tax equilibria

If $\theta > \beta + \delta$, the right-hand side of (10) is bounded away from zero. As $\tau \to 0$, therefore, the left-hand-side must converge to a non-zero limit, which implies that $\bar{m} = \bar{\tau}/\rho$ converges to a non-zero limit. This limiting no-tax equilibrium is analogous to the "nice" no-tax equilibrium in the Samuelson model. However, if there is actually no tax, non-zero storage is possible, and there are other equilibria and initial price levels. For example, if storage persists forever, then the inflation rate must be $-\gamma$ forever, and since $\bar{M}$ must be constant when there is no tax or transfer, $\bar{m} \to 0$ as $t \to \infty$. Any initial level of $\bar{m}$ that is small enough will correspond to an equilibrium in which people make savings decisions consistent with the $\rho = \gamma < 0$ rate of return, but with earlier generations, which have larger m_0 endowments, consuming at a higher rate than the steady-state no-storage level. If the initial price level is low enough, and therefore initial (in absolute time) m_0 is high enough, the elevated level of consumption from the initial endowments of m_0 may be high enough that aggregate storage becomes zero in finite time. At that point, the economy could switch over to the "nice" no-tax equilibrium, but to avoid a jump in prices (ruled out by no-arbitrage) at that point, it would have to be that the level of $\bar{m}$ at that point exactly matches the value associated with the nice no-tax equilibrium. Since $\bar{m}$ simply shrinks at the rate γ before that point, there is a unique initial price level associated with this equilibrium in which storage vanishes in finite time. Higher initial price implies storage persists forever an the economy converges to barter equilibrium; lower prices are ruled out by the arbitrage opportunity they imply at the date where storage has disappeared.

V. What if τ is constant across age?

In the original problem statement τ did not depend on age. This case leads to $\rho = \beta + \delta - \theta$ in equilibrium, so that the rate of return does not depend on τ. For there to be an equilibrium with positive ρ, then, the endowment must decay quite slowly, and in fact the requirement is the opposite of what was required to allow a no-tax steady state with valued money that we studied in the previous section. This is because with constant τ, the old hold large amounts of real balances which they use almost entirely to pay taxes, so the young require an incentive to dissave the large real balances they inherit from the old. An equilibrium with negative ρ is not possible with $\tau > 0$, because the the government budget constraint implies that in that case real balances disappear in finite time.

VI. How to start from $t = 0$

Finally, how can tax-backed money be introduced into this economy at some initial date $t = 0$, when up to then the economy has had positive storage? Money would be introduced via lump-sum discrete transfers to existing people of all ages. The storage cannot be eaten up instantly, though, so there will be at least some time during which storage exists and the inflation rate is therefore positive at $-\gamma$. During this time, real balances will shrink, but so will storage if the real balances are high enough (and people therefore feel rich enough). When storage disappears, real balances would then have to match the level $\bar{m}$ corresponding to the tax-backed steady state we have worked out above. This would in turn require that the initial price level be at exactly the value that makes $e^{\gamma T}\bar{M}_0/P_0 = \bar{m}$, where T is the amount of time until storage has disappeared.