BUBBLE MONEY IN CONTINUOUS TIME EXERCISE

In this exercise you try to construct and analyze a model that captures the main ideas of the two-period-lived Samuelson model of bubble money, but in continuous time.

The model has a population of agents that at first we will take to be fixed in size. Agents of all ages die at the rate δ at all times, so the fraction of the population born at t that is still alive at $t+s$ is $e^{-\delta s}$. Each agent has an endowment stream that shrinks over time at the rate θ. That is the agent's age-t endowment is $Y_0 e^{-\theta t}$. All agents pay the same tax τ each period of their life; the "tax" can be negative, i.e. could be a transfer payment. The government uses the tax to retire (or expand, if $\tau < 0$)) the supply of fiat money. People can save or dissave from their money balances or from their stock of savings. The savings pay a rate of return γ. If we are going to mimic the Samuelson results, we will want $\gamma < 0$ in this constant-population case. Negative money balances or savings are impossible. The wealth of the dying agents in the form of real savings or money balances is distributed by the government equally among all newly born agents.

The optimization problem of the individual agent is

$$\max_{C,S,M} \int_0^\infty e^{-(\beta+\delta)t} U(C_t)\, dt \quad \text{subject to} \tag{1}$$

$$C + \dot{S} + \frac{\dot{M}}{P} = Y_0 e^{-\theta t} + \gamma S - \tau \tag{2}$$

$$M > 0\,, \qquad S > 0\,, \tag{3}$$

where β is the individual's discount rate, P is the price level, and t is the individual's age. It is up to you to formulate the government budget constraint (bearing in mind that M will be different for people of different ages). Assume $U(C) = \log(C)$ in the exercises below.

(1) Display the government budget constraint.

(2) Suppose money is valueless and $\tau = 0$, so the only government activity is redistributing any S left by the dying to the newborns. What is the time path of individual consumptions in that case, and how does it depend on β, γ and θ and δ?

(3) Is there an equilibrium in which agents of a given vintage always choose the same lifetime pattern of C, M and S and in which aggregate M is constant? Are there many such equilibria? How does the answer depend on β, γ, θ and δ?

(4) Does this model generate a result like that in the Samuelson model, where any negative value for τ implies that the equilibrium is unique, and better than that with valueless money?

(5) What do you think would happen to your conclusions if population were growing at a constant rate? What if it were known to be growing for some time, then shrinking?

(6) Would it make a difference to the model's behavior if we introduced the possibility of borrowing and lending between individuals?