## TAKEHOME MIDTERM EXAM

(1) Consider an economy where the representative agent solves

$$
\begin{gather*}
\max \int_{0}^{\infty} e^{-\beta t} \log \left(C_{t}\right) d t \quad \text { subject to }  \tag{1}\\
C \cdot\left(1+\gamma v_{t}\right)+\frac{Q \dot{X}+\dot{M}}{P}+\dot{F}=\bar{Y}+\rho F+\frac{X}{P}-\tau  \tag{2}\\
v=\frac{P C}{M}  \tag{3}\\
X \geq 0  \tag{4}\\
F_{t} e^{-\rho t} \xrightarrow[t \rightarrow \infty]{ } 0 . \tag{5}
\end{gather*}
$$

Here $C$ is consumption, $v$ is the velocity of money, $\gamma v$ is the fraction of consumption lost to transactions costs, $Q$ is the dollar price of a consol, $X$ is the number of consols held, $M$ is currency, $P$ is the price level, $F$ is net holdings of the foreign asset, $\rho$ is the real rate of return on the foreign asset, $\bar{Y}$ is the level of the constant endowment stream, and $\tau$ is lump-sum taxes. That equation (5)'s left-hand side does not have a negative limit is the foreign lenders' transversality condition, while that it does not have a positive limit is a consequence of the agent's transversality condition.

The government faces the constraint

$$
\begin{equation*}
\frac{Q \dot{X}+\dot{M}}{P}=\frac{X}{P}-\tau \tag{6}
\end{equation*}
$$

Its policy is to set $Q \equiv \bar{Q}$ and $\tau \equiv \bar{\tau}$, i.e. to make both the price of consols and the level of lump sum real taxes constant.

Note that the government does not hold $F$ assets and (implicitly, because we use the same letters for these variable in government and private constraints) only domestic agents, not foreigners, hold the consol debt. In what follows assume initial $X, M$, and $F$ are all strictly positive.
(a) Display the economy's social resource constraint and the first-order conditions for an optimum in the private agent's problem.
Subtracting the government budget constraint from the agent's budget constraint gives us the social resource constraint:

$$
C+\dot{F}=\bar{Y}+\rho F
$$

The FOC's are

$$
\begin{array}{ll}
\partial C: & \frac{1}{C}=\lambda \cdot(1+2 \gamma v) \\
\partial M: & -\frac{\dot{\lambda}}{P}+\frac{\beta \lambda}{P}+\frac{\lambda \dot{P}}{P} \frac{\bar{P}}{P}=\frac{\lambda}{P} \gamma v^{2} \\
\partial X: & -\frac{\dot{\lambda} Q}{P}+\frac{\beta Q \lambda}{P}-\frac{\lambda \dot{Q}}{P}+\frac{\lambda Q}{P} \frac{\dot{P}}{P}=\frac{\lambda}{P} \\
\partial F: & -\dot{\lambda}+\beta \lambda=\lambda \rho .
\end{array}
$$

(b) Suppose the economy has been in an equilibrium with $\rho=\beta$, both constant, and $\bar{Q}=\beta^{-1}$. Agents are assumed to have perfect foresight. Characterize this equilibrium. In particular, state which variables are constant, which changing, and how fast the changing variables are growing or shrinking? Is this equilibrium unique?
The FOC's can be rewritten as

$$
\begin{align*}
\frac{1}{C}=\lambda \cdot(1+2 \gamma v) & \\
\frac{-\dot{\lambda}}{\lambda} & =\rho-\beta \\
\frac{\dot{Q}}{Q}+\frac{1}{Q} & =\gamma v^{2}  \tag{*}\\
\rho & =\frac{\dot{Q}}{Q}+\frac{1}{Q}-\frac{\dot{P}}{P} \tag{†}
\end{align*}
$$

With $\rho=\beta$ and $\bar{Q}=\beta^{-1}$. these imply that $v, C$ and $P$ are constant. The government budget constraint can be rewritten, using the FOC's, as

$$
\frac{d}{d t}\left(\frac{Q X}{P}\right)=\rho \frac{Q X}{P}-\tau-\frac{\dot{M}}{P}
$$

which can be solved forward, suing private sector transversality to eliminate explosive paths for real government debt, to deliver

$$
\frac{Q X}{P}=\frac{\tau}{\rho}+\int_{0}^{\infty} e^{-\rho s} \frac{\dot{M}_{t+s}}{P_{t+s}} d t
$$

Because we have already concluded that $v, C$, and $P$ are constant, $M$ is constant and it must be therefore that

$$
\frac{\bar{Q} X}{P}=\frac{\tau}{\rho} .
$$

With $C$ constant, the social resource constraint can be solved forward to yield the unique value $C=\rho F+\bar{Y} . M / P$ therefore also uniquely determined (as well as
constant), because $v$ is determined by the fixed- $Q$ policy and (*). Rearranging the budget constraint, without using any forward-looking FOC's, gives us

$$
\frac{d}{d t}\left(\frac{\bar{Q} X}{P}\right)=\frac{X}{P}-\bar{\tau}-\frac{\dot{M}}{P}-\frac{\bar{Q} X \dot{P}}{P} \frac{\dot{P}}{P}
$$

The left-hand side of this equation must be, we have concluded, zero. The last two terms on the right-hand side are also zero, because $M$ and $P$ are constant on the equilibrium path. We must therefore have $X / P=\tau$. $X$ is not given by history: $X$ and $M$ can jump discontinuously at the initial date so long as $B_{0}=\bar{Q} X+M$ does not jump. This follows because the government budget constraint is not forwardlooking and $\dot{X}$ and $\dot{M}$ appear in it as the linear combinations $\bar{Q} \dot{X}+\dot{M}$. But since $m_{0}=M / P$ is also uniquely determined,

$$
\frac{B_{0}}{P}=\frac{\bar{Q} X+M}{P}=\frac{\tau}{\rho}+m_{0}
$$

there is only one value of initial $P$ that satisfies the flow budget constraint, and the equilibrium is unique.
(c) Now suppose that, at time $t=0, \rho$ increases to a value above $\beta$. This change was not anticipated, but now that it has occurred the agents in the economy know that it will persist forever. Monetary and fiscal policy remains unchanged, with the same levels of $\bar{Q}$ and $\bar{\tau}$ as before. Which variables, if any, will change discontinuously at time 0 , and in which direction will they jump, and which variables will remain constant? After $t=0$, which variables are constant, which changing, and how fast are the changing variables growing or shrinking?
Since $Q$ is still constant at $\beta^{-1}, v$ is still constant, and at the same value, across the change in $\rho$. But because now $\rho>\beta,-\dot{\lambda} / \lambda=\dot{C} / \dot{C}$ is positive and equal to $\rho-\beta$. Solving the social resource constraint forward now tells us

$$
C_{0}=\beta F_{0}+\frac{\bar{Y} \beta}{\rho} .
$$

The increase in $\rho$ will therefore produce a downward jump in $C$, followed by exponential growth at the rate $\rho-\beta$. Since there is no change in velocity, real balances jump downward initially also, followed by exponential growth in real balances at the rate $\rho-\beta$. Since $\bar{Q}$ remains unchanged, $\dagger$ implies that $\dot{P} / P=\beta-\rho$, and therefore, to keep velocity constant, that $M$ is constant. With $M$ constant, there is no seigniorage, so once again we conclude that $\bar{Q} X / P=\bar{\tau} / \rho$. This means the real value of interest-bearing debt jumps downward when $\rho$ jumps upward. So we have determined that both $M / P$ and $\bar{Q} X / P$ jump downward when $\rho$ increases. Since $\bar{Q} X+M$ cannot jump, this means $P$ must jump upward. The path of $C$ is then a downward jump followed by exponential growth, while that of $P$ is an upward jump followed by deflation. Nominal consumption $P C$ remains constant. Because after the initial date $P$ is decreasing, while $\bar{Q} X / P$ must remain constant,
the number of consols $X$ must shrink to offset the decline in $P$, meaning the usual nominal government deficit concept ( the net new issuance of nominal debt) is negative.
(2) A seller has a different cost every day, but it only takes on two values, "high" or "low". She also can only choose between two values of price, again "high" or "low". Her profits are described by this table:

|  | cost |  |
| ---: | :---: | :---: |
| price | low | high |
| low | 6 | 0 |
| high | 3 | 2 |

The cost is high or low with equal probability, but the seller has the option of collecting information so that her conditional distribution each day is more accurate than the marginal equal-probability distribution. We assume that there is a cost to collecting such information that is linear in the Shannon mutual information between cost and price.
(a) Is there a cost of information such that the seller would always choose the low price? Is there a cost of information such that the seller would always choose the high price?
If the cost of information is high enough, the agent will collect none, simply using the equal unconditional probabilities. At those probabilities, the low price offers expected profit 3 , while the high price delivers profit 2.5 , so the agent would always choose the low price. No cost of information could lead to always selecting the high price, since with that pricing decision rule expected returns are 2.5 , and always choosing the low price delivers higher profits than that, with zero cost of information. A complete answer might have gone on to determine the critical level of information cost above which no information is collected. This is around 6.1 per bit.
(b) The worst outcome is obviously low price, high cost. Is it possible that the seller, for some cost per unit of information, would choose a signal such that the probability of the low price, high cost event is zero, while all the other three have non-zero probability? Explain your answer, or else give an example showing that it is possible.
This can't happen. It implies that, conditional on price being low, the probability of cost being low is one, so that the entropy of the conditional distribution for cost given low price is zero. For a distribution concentrated on two points with probabilities $p$ and $1-p$, the derivative of entropy with respect to $p$ is $-\log (p /(1-p))$, which is $-\infty$ at $p=0$. This means that a small increase above zero in the probability of high cost given low price decreases information costs at an arbitrarily high rate, while it clearly reduces expected profits at a finite rate. Only if the solution made the probability of low price zero, so that the conditional distribution of cost
given low price did not affect information costs, could the probability of low price and high cost be zero.
(c) There is a critical cost per bit of information such that for any cost below that level, the solution gives the two diagonal cells, high-high and low-low, total probability one. What is this critical value?
This isn't true, actually. The same argument given under part 2aimplies that, if the diagonal probabilities are positive, so are the off-diagonal probabilities. Nonetheless, when the cost of information gets down to around .13, the solution is so concentrated on the diagonal that the off-diagonal probabilities are zero to within computational rounding error.
(d) Display a solution to the problem for some information cost such that all four cells have positive probability - or else show that there are no such solutions. This last question is I think doable with a calculator in reasonable time. If you can't see how to get a numerical solution in reasonable time, you can get substantial credit for displaying equations and constraints that could be passed to a computer for solution with a constrained optimization routine. Here's a solution with $\theta=2$.

$$
\begin{gathered}
p(\text { low price })=.6474 \\
p(\text { low cost } \mid \text { low price })=.6886 \\
p(\text { high cost } \mid \text { high price }=.8463
\end{gathered}
$$

Though you could do this with a calculator, I used this R function:

```
thinfo <- function(theta) {
    umat <- matrix(c(exp(6/theta), exp(3/theta), 1, exp(2/theta)), 2,2)
    h <- solve(umat, c(1,1))
    p <-solve (t (umat), 1/(2 * h))
    f <- (p %o% h) * umat
    return(list(h=h, p=p, f=f))
    }
```

The program uses the first order conditions for an optimum to solve for the probabilities of high and low price. When the cost of information (theta) gets too high, the solution makes one element of the calculated probability vector negative. The returned $p$ is the probabilities of the prices, the matrix $f$ is the joint probability distribution, and $h$ is the vector of Lagrange multipliers on the adding-up constraints. The elements of $h$ must also be non-negative if the program output is a solution, though in this example that possibility does not arise.

