EXERCISE: PRICE DETERMINATION WITH CORRIDOR POLICY

Consider a model in which the representative agent solves

$$\max_{C,B,M} \int_0^\infty e^{-\beta t} \log(C_t)\, dt \tag{1}$$

subject to

$$C \cdot (1 + f(v)) + \frac{\dot{B} + \dot{M}}{P} = \frac{rB + qM}{P} + Y - \tau \tag{2}$$

Here C is consumption, B is duration-zero government debt, M is reserves, P is the price level, r is the interest rate on government debt, q is the interest rate on reserves, Y is the flow rate of endowment, τ is the rate of lump sum taxation, and $v = PC/M$ is velocity. Assume $f(v) = \psi_0 \exp(-\psi_1/v)$.

The government has the budget constraint

$$\frac{\dot{B} + \dot{M}}{P} = \frac{rB + qM}{P} - \tau\,. \tag{3}$$

Here we interpret M as reserves rather than as currency. M holdings pay interest but at a rate q that is smaller (because M provides transactions services) than the rate r that applies to ordinary government debt. Because there is no currency, both r and q could possibly become negative.

(1) Show that if policy fixes r, q and τ at constant values, there is a uniquely determined equilibrium for both real and nominal variables. How do steady-state C and steady-state $\pi = \dot{P}/P$ depend on r, q and τ?

The FOC's of the agent are:

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t \cdot (1 + f(v) + v f'(v)) \tag{A1}$$

$$\partial B: \qquad -\frac{\dot{\lambda}}{P} + \beta\frac{\lambda}{P} + \frac{\lambda}{P}\frac{\dot{P}}{P} = r\frac{\lambda}{P} \tag{A2}$$

$$\partial M: \qquad -\frac{\lambda}{P} f'(v) v^2 - \frac{\dot{\lambda}}{P} + \beta\frac{\lambda}{P} + \frac{\lambda}{P}\frac{\dot{P}}{P} = q\frac{\lambda}{P}\,. \tag{A3}$$

These imply

$$f'(v) v^2 = r - q\,, \tag{$*$}$$

so long as $\lambda > 0$ (i.e. the budget constraint binds.) So if r and q are fixed, v is fixed. Of course there are limits on feasible choices of r and q. Since v and $f'(v)$ are naturally assumed non-negative, we need $r > q$, and if $f'(v)v^2$ is bounded, that puts a bound on $r - q$. For the f you were to assume,

$$f'(v) v^2 = \psi_1 \psi_0 \exp(-\psi_1/v) < \psi_1 \psi_0\,. \tag{A4}$$

As should have been explicitly stated, you can assume Y is constant. Then the social resource constraint (private constraint minus government budget constraint) and constant velocity implies C is constant, which in turn implies, via the C FOC, that λ is constant. Using this in the B FOC, we get that $r = \beta + \pi$. Then we can solve $(*)$ for q. Since we know C and v at this point, we know $M/P = m = C/v$. To keep v constant, we must have $\dot{M}/M = \pi = r - \beta$. The government budget constraint in real terms, using what we have already deduced about the equilibrium, is

$$\dot{b} = \beta b - \pi m - \tau\,. \tag{A5}$$

We've used the conditions $r = \pi + \beta$ and can write the seigniorage term $\dot{M}/P$ as πm because we know m and π are determined and constant. This government budget constraint is an unstable differential equation in b, with a unique steady state at $b = (\tau + \pi m)/\beta$. Since at the initial date $B + M$ (not B and M separately) is given by history, and since we now have determined unique initial values for b and m, we can find the initial price level from $B + M = (b + m)P$. Unstable solutions to the budget constraint that explode upward are ruled out by transversality ($(b + m)\lambda e^{-\beta t} \to 0$) and downward exploding solutions are ruled out by the impossibility of agents holding negative money or borrowing without limit from the government.

Steady-state C is simply Y diminished by transactions costs, so it starts at Y for $v = 0$ (assuming $f(0) = 0$) and shrinks as v increases, and thus shrinks with r increases but grows with q increases. With the problem's specific $f(v)$, there is a lower bound of $Y/(1 + \psi_0)$ on C. Inflation is just $r - \beta$, so does not depend on q. τ affects neither C nor π.

(2) Suppose instead policy sets $q = r - \delta$, where δ is constant, and fixes the money stock at $M_t \equiv \bar{M}$, while again τ is held constant. Is there a uniquely determined time path of prices in this case? If there are many, is there just one path of prices that is non-explosive? (A constant inflation rate counts as non-explosive.)

Here again the policy pegs v via $(*)$, so again C is constant and $r = \beta + \pi$. The constant M policy implies, together with the constant C and v, that P is constant and therefore $\pi = 0$. But we get a specific solution for P here, without having used the fiscal policy or government budget constraint. Since M is constant and $\pi = 0$ there is no seigniorage, and the government budget constraint is

$$\dot{b} = \beta b - \tau. \tag{A6}$$

This is an unstable differential equation whose solution explodes up or down unless $b \equiv \tau/\beta$. Transversality of the private agent is the condition $b_t \lambda_t e^{-\beta t} \to 0$ as $t \to \infty$. This rules out upward exploding solutions of the government budget constraint, and downward explosive ones are ruled out by a requirement that the public cannot borrow in unlimited amounts from the government. Since B is given by history at the initial date, the unique possible value for b gives us a second way to solve for initial P, which in general conflicts with the P we found above without using the government budget constraint. So this policy combination allows for no equilibrium, except in the knife-edge special case where the solution to $P = vC/M$ happens to match $P = \beta B_0/\tau$, where B_0 is the initial level of the interest-bearing debt.

(3) Suppose as yet another case that policy sets $M_t \equiv \bar{M}$ as above but sets $q = \delta r$, $0 < \delta < 1$. τ still constant. Again, is there a unique equilibrium? If there are many, is there just one that's not explosive? For this case only (and below when

reconsidering it with a Taylor rule), change the agents' budget constraint to

$$C \cdot (1 + f(v)) + \frac{\dot{B} + \dot{M}}{P} + \dot{F} = \frac{rB + qM}{P} + \rho F + Y - \tau \tag{4}$$

and assume ρ is constant. This implies that agents can invest in an asset F with fixed real rate of return ρ. It changes the social resource constraint. It makes solution much easier for this case.

The addition of F to the list of decision variables gives us an additional FOC:

$$\partial F : \qquad -\dot{\lambda} + \beta\lambda = \lambda\rho\,. \tag{A7}$$

We can use this, together with the B FOC, to arrive at

$$r = \rho + \pi\,. \tag{A8}$$

Unlike the steady-state condition $r = \beta + \pi$ that we had above, this one holds without requiring v or C to be constant. Constant v no longer implies constant C, because agents can borrow and invest externally via F.

From the B and M FOC's we get

$$f'(v)v^2 = (1 - \delta)r\,. \tag{A9}$$

From the C and F FOC's we get

$$\frac{\dot{C}}{C} + \frac{2f'(v) + f''(v)v}{1 + f(v) + f'(v)v}\dot{v} = \rho - \beta\,. \tag{A10}$$

And the definition of velocity implies

$$\pi = \frac{\dot{v}}{v} - \frac{\dot{c}}{c}\,. \tag{A11}$$

Using the four equations (A8-A11), we can derive a differential equation in v alone:

$$\frac{\dot{v}}{v}\left(1 + \frac{2f'v + f''v^2}{1 + f + f'v}\right) = \frac{f'v^2}{1 - \delta} - \beta\,. \tag{A12}$$

This equation is easily seen to have a steady state at the value of v satisfying $f'v^2 = \psi_0\psi_1 e^{-\psi 1/v} = (1 - \delta)\beta$, if such a value of v exists. However, even if such a value of v exists, it is not generally consistent with equilibrium. With our choice of f, $f'v^2 = \psi_0\psi_1 e^{-\psi_1/v}$ is increasing in v, so for any v above the steady state value, v is increasing. It approaches an upper bound of $\psi_0\psi_1/(1 - \delta)$ if it starts above the steady state value and a lower bound of $-\beta$ if it starts below.

The level of M is fixed by policy, and P, since there is no seigniorage, is fixed by the requirement that $b = \tau/\rho$ as in the previous section. Only one value of C can be consistent with these values of v, M and P. But we can solve the social resource forward to get another uniquely determined value of C. Given any initial value of v, the fact that initial M and P are fixed implies that initial C is determined by the initial v. Indeed $v = PC/M$ prescribes a linear relation between initial v and C. From then on, we can find the path of v from (A12), and then from (A10) and the initial value of C find the time path for C. But then the forward solution of the social resource constraint tells us

$$F_0 + \frac{Y}{\rho} = \int_0^\infty e^{-\rho t}(C_t \cdot (1 + f(v_t))\, dt\,. \tag{A13}$$

F_0 is given by history. (It might seem that individuals could rebalance portfolios by trading B and M for F, but we assume that the government does not hold F and foreigners do not sell or buy B or M. Or, if we interpret F as a domestic constant-returns investment opportunity, we cannot instantly increase or decrease F, instead only accumulating it or decumulating through savings or dissavings.) Each initial v_0 determines a unique value for the right-hand-side of (A13). Probably the right-hand-side is monotone in initial v, though this seems hard to prove. (If not, there might be multiple real equilibria.) That it is monotone seems likely because it can be seen from (A10) that $\dot{C}/C$ converges to $\rho - \delta$ both as $v \to \infty$ and as $v \to 0$. In any case, whatever value or values of C_0 emerge from (A13) are unlikely to match the unique C_0 consistent with the stable solution. So the solution, if it exists, will, except in knife-edge cases, be one of the unstable paths of v that solve (A12).

These unstable paths will violate no transversality or feasibility condition. Indeed, in solving forward to get (A13), we used the transversality condition and the condition that unbounded borrowing (negative F) from abroad is impossible. C will follow a reasonable path, indeed will converge to a constant if $\rho = \beta$, for any equilibrium path of v.

I was mistaken in thinking that introducing F simplified this problem. It simplifies the Taylor-rule policy version of the problem, but not this one.

(4) Suppose fiscal policy is passive, i.e. $\tau + \dot{M}/P = -\phi_0 + \phi_1 B/P$ with $\phi_1 > \beta$. Determine whether there is a unique equilibrium, and if not whether there is only one stable equilibrium, for the same three monetary policies considered above (fixed r and q, fixed M with $q = r - \delta$ or with $q = \delta r$.)

In the first problem, we used the active fiscal policy to pin down the price level by making the government budget constraint unstable. With passive fiscal policy, we can't do this, so the price level is indeterminate.

In the second problem the generic non-existence came from using the active fiscal policy to find an initial P different from the one we could derive from the rest of the model. So the passive fiscal policy makes an equilibrium exist in this case, and it might appear unique, despite the passive fiscal policy. However there is a major qualification. Deriving $(*)$ from the FOC's assumes $\lambda/P > 0$. λ will certainly be positive always, but $P = \infty$, i.e. valueless money, is possible. With $P = \infty$, the B and M FOC's are trivially satisfied, so there is always a second equilibrium with valueless money. This was ruled out above when we had active fiscal policy, but with passive fiscal policy, the valueless money solution is possible. We have to reinterpret fiscal policy, though. The usual simple passive rule, $\tau = -\phi_0 + \phi_1 b$, implies $\tau = -\phi_0$ at $b = 0$ and thus that $\dot{b}$ is positive. But with money valueless, it is impossible for the government to make $\dot{b} > 0$ if it only issues nominal debt. We need, therefore, to specify a feasible fiscal policy for the $P = \infty$ case. One possibility is that with $P = \infty$ the government switches to real bonds. That is feasible, and results in valueless money being a viable equilibrium. A switch to active fiscal policy whenever $P = \infty$ is feasible, but it makes the $P = \infty$ case impossible, since then the P consistent with the fiscal policy must be positive, and, as in the version of this case with fiscal policy always active, generally inconsistent with the P emerging from the other equations of the system. So we can arrive at a unique equilibrium here by combining the stated policies with a switch to active fiscal policy at $P = \infty$.

In the third problem we found a unique equilibrium (when it existed) with $v \to 0$ or $v \to \infty$ by using the active fiscal rule. Without the active fiscal rule, any initial price level and initial v is possible. There are infinitely many equilibria, all of them consistent with feasibility and transversality, but only one has v bounded away from zero and infinity.

Extra credit, or food for thought: Consider what happens when the fixed-M or fixed-r policies considered above are replaced by an interest-smoothing Taylor rule:

$$\dot{r} = \theta_r\big(\theta_\pi(\pi - \bar{\pi}) - (r - \bar{r})\big)\,, \qquad \theta_\pi > 1 \qquad \theta_r > 0 \tag{5}$$