# EXERCISE: SOLVING A NONLINEAR DYNAMIC EQUILIBRIUM MODEL

Consider an economy in which nothing interesting is going on except variation in fiscal policy. That is, the representative individual sees himself as facing the optimization problem

$$\max_{C,B} E\left[\sum_{t=0}^{\infty} \beta^t \log C_t\right] \quad \text{subject to} \tag{1}$$

$$C_t + \frac{B_t}{P_t} = R_{t-1}\frac{B_{t-1}}{P_t} - \tau_t + Y_t\,, \tag{2}$$

while the government faces the budget constraint

$$\frac{B_t}{P_t} = R_{t-1}\frac{B_{t-1}}{P_t} - \tau_t\,. \tag{3}$$

This of course means that $C_t \equiv Y_t$, and we will assume $Y_t$ is constant at $Y_t = Y$, so consumption in equilibrium does not vary at all, even though agents see themselves as able to trade off consumption at different dates via the government bond market. Note that the model can be rewritten in terms of $b_t = B_t/P_t$ and $\pi_t = P_t/P_{t-1}$ in place of $B_t$ and $P_t$.

Government fiscal and monetary policy is

$$R_t \equiv \beta^{-1} \tag{4}$$

$$\tau_t = \frac{e^{.4b_t}}{100 + e^{.4b_t}} + g_t\,. \tag{5}$$

$g_t$ is an exogenous Markov process, equal to 0 or 1. The transitions between states for $g_t$ are governed by the transition matrix

$$P[S_{t+1} = i \mid S_t = j] = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}\,. \tag{6}$$

We assume the agents always know the current value of $g_t$. Assume $\beta = .95$.

If it we had simply $\tau_t = g_t$ as fiscal policy, this would be a standard active-fiscal, passive money model. The current real value of the debt would be the expected discounted value of future $g_t$, which could be calculated directly as a function of current $g_t$. Surprise inflation and deflation would keep the real value of the debt at these values.

In this model, though, there are three deterministic steady states for the debt, and primary surplus is an increasing function of the level of debt. If it were a linearly increasing function of the debt over all debt levels, no matter how large or small, the budget constraint would imply stationary behavior for the real debt and equilibrium would be indeterminate. Here, though, there is what Leeper calls a "fiscal limit" — at large values of debt the primary surplus starts increasing very weakly with debt, and indeed the primary surplus is bounded above. It is also bounded below as debt gets smaller.

Because of these bounds, there are upper and lower bounds on $b$ itself. Define the function $\phi(b) = \frac{e^{.4b}}{100+e^{.4b}}$. Note that $E_t g_{t+1} = .1 + .8g_t$. Then $\phi(b)$ is bounded above by 1 by construction, and the discounted present value of primary surpluses is less than or equal to

$$\frac{1.5\beta}{1-\beta} + \frac{.8(g_t - .5)\beta}{1-.8\beta},$$

which is therefore an upper bound on $b$ — with distinct values for $g = 1$ and $g = 0$. A similar calculation yields a pair of lower bounds on $b$, because $\phi()$ is also bounded below. The system has two state variables, $b_{t-1}$ and $g_t$. The two endogenous variables $b$ and $\pi$ will each be functions of these two state variables. Given the form of either $b(b_{t-1}, g_t)$ or $\pi(b_{t-1}, g_t)$, the other one can be derived from the government budget constraint.

The stability of the system is enforced by using knowledge of the bounds on $b$ — you should use a from for the two $b(\cdot, g)$ functions that never goes outside the bounds. (Since initial $b_{-1}$ might be outside the bounds at time zero, the function's domain can go outside the bounds; it's only from time $t = 0$ onwards that the bounds hold.) Also note that bounds calculated as I did above are likely not sharp — there may in fact be tighter bounds. But the tighter bounds should emerge from your solution. So long as you enforce bounds that avoid explosive solutions for $b$, your solution should not be sensitive to exactly what bounds your functional form imposes.

This is a problem that can be solved with projection methods or the like.

What distinguishes it from the problem and "solution" I presented in class Thursday? One difference, that is actually inessential, is that now $R$ is constant. This just eliminates the non-uniqueness in the inflation path. The solution for the $b$ path is essentially the same. My mistake in what I did on the board in class was to assume that $b$ would emerge as a function of current $g_t$ alone. In fact, in this model, $b$ may stay a long time in the neighborhood of the deterministic steady state, because the model is locally stable there. In this region, inflation may show little variation and debt will simply look mean-reverting. But a long sequence of $g_t = 0$ realizations will drive debt up in the absence of much surprise inflation, and as debt approaches its upper limit, inflation must get more variable. It must be expected that yet another $g_t = 0$ realization will be accompanied by inflation to keep $b_t$ from going over the limit, which entails (via the FOC) that a $g_t = 1$ realization will be accompanied by deflation.

It is still possible that there is no equilibrium in this setup — that the fiscal rule is infeasible. Nonetheless it is possible to set up a class of candidate $b()$ or $\pi()$ functions, specify an accuracy criterion, and proceed to try to find an accurate solution. You can get full credit if you set up an accuracy criterion and a reasonable class of candidate $b()$ or $\pi()$ functions, with accompanying code. Actually achieving convergence is not required. Of course you can also get full credit by analytically deriving the form of the equilibrium or proving that there is no equilibrium.