## TAKE-HOME MIDTERM EXAM

Answer both questions. Unlike problem sets, where collaboration is encouraged, on a take-home exam collaboration is not allowed. Do not discuss the exam with other people until after the time it is due, 9AM Friday, 10/29
(1) Rational inattention and adjustment costs are often thought of as alternative ways to introduce frictions and sluggish adjustment into economic models. Suppose we are considering two models of the behavior of agents who are solving a tracking problem. The agents are trying to minimize the discounted sum of squared deviations of $x_{t}$ (their choice variable) from $z_{t}$ (an exogenously given stochastic process). The process $z_{t}$ satisfies

$$
\begin{equation*}
z_{t}=\rho z_{t-1}+\varepsilon_{t} \tag{1}
\end{equation*}
$$

with $\rho \in(0,1)$ and $\varepsilon_{t}$ i.i.d., $N\left(0, \sigma^{2}\right)$, and independent of all past values of $z$. The two models postulate two different types of adjustment costs, with the objective functions specified below.

$$
\begin{array}{ll}
\min E\left[\sum_{t=0}^{\infty} \beta^{t}\left(\left(x_{t}-z_{t}\right)^{2}+\theta\left(x_{t}-x_{t-1}\right)^{2}\right)\right] & \text { adjustment costs } \\
\min E\left[\sum_{t=0}^{\infty} \beta^{t}\left(\left(x_{t}-z_{t}\right)^{2}+\phi I\left(x_{t}, z_{t} \mid\left\{x_{s}, s<t\right\}\right)\right)\right] & \text { rational inattention } \tag{3}
\end{array}
$$

In the rational inattention objective function, $I\left(x_{t}, z_{t} \mid\left\{x_{s}, s<t\right\}\right)$ is the mutual information between $x_{t}$ and $z_{t}$ in their joint distribution conditional on $x$ values before time $t$.
(a) Say what you can about the nature of the two solutions. You should be able to solve the adjustment cost problem explicitly. You may be able to get an explicit solution to the rational inattention problem by following the assigned readings, but do not spend disproportionate time working on this. Assume without proving it that the solution makes everything jointly normal in the RI version of the problem.
The adjustment cost problem has first-order condition

$$
x_{t}-z_{t}+\theta\left(x_{t}-x_{t-1}\right)-2 \beta \theta E_{t}\left[x_{t+1}-x_{t}\right]=0 .
$$

This leads to

$$
\left(-\theta \beta L^{-1}+1+(1+\beta) \theta-\theta L\right) x_{t}=z_{t} .
$$

The product of the roots of the polynomial in $L^{-1}$ is $1 / \beta$, and for reasonable parameter values both roots are positive, with one bigger than one and one
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less than one. Letting the smaller and larger roots be $r_{1}$ and $r_{2}$, respectively, this leads to a solution of the form

$$
x_{t}=r_{1} x_{t-1}+\frac{E_{t} \sum_{s=0}^{\infty} r_{2}^{s} z_{t+s}}{1+\theta(1+\beta)}=\frac{z_{t}}{\left(1-\rho r_{2}\right)(1+\theta(1+\beta))} .
$$

The important point about this solution for the rest of the answer is that it makes $x_{t}$ an exact linear function of $z_{t}$ and $x_{t-1}$, with no random disturbance. In the RI problem, once we have accepted that everything is going to be joint normal, we can reformulate by letting $\hat{z}_{t}$ denote the mean of the distribution of $z_{t}$ based on information up through time $t$. Then certainty-equivalence tells us that it will be optimal to set $x_{t}=\hat{z}_{t}$. If we let $\hat{\sigma}_{t}^{2}$ be the variance of the distribution of $z_{t}$ conditional on information up through time $t$, we can rewrite the problem as

$$
\min E\left[\sum_{t=0}^{\infty} \beta^{t}\left(\hat{\sigma}_{t}^{2}+\phi \frac{1}{2} \log \left(\frac{\rho^{2} \hat{\sigma}_{t-1}^{2}+\sigma_{\varepsilon}^{2}}{\hat{\sigma}_{t}^{2}}\right)\right)\right] .
$$

The term in the numerator is the variance of the distribution of $x_{t}$ based on information through time $t-1$, which is $N\left(\rho \hat{z}_{t}, \rho^{2} \hat{\sigma}_{t-1}^{2}\right)$. We have used the fact that the mutual information between two normal random variables is half the $\log$ of the ratio of the unconditional variance of one of them to its conditional variance given the other one.
The FOC for this reformulated problem, in which the $\hat{\sigma}_{t}^{2}$ sequence is all that is chosen, is

$$
1+\beta \phi \frac{\rho^{2}}{\hat{\sigma}_{t}^{2} \rho^{2}+\sigma_{\varepsilon}^{2}}=\frac{\phi}{\hat{\sigma}_{t}^{2}} .
$$

This has at least one positive root, as can be seen from the fact that as $\hat{\sigma}_{t}^{2} \rightarrow \infty$ the right-hand-side goes to zero and the left-hand side to one, whereas as $\hat{\sigma}_{t}^{2} \rightarrow 0$ the right-hand-side goes to $+\infty$ while the left-hand side goes to a finite number. So, because the left and right-hand sides are both continuous, There is at least one value of $\hat{\sigma}_{t}^{2}>0$ that solves the equation. Multiplying through by the product of the denominators in the FOC gives us a quadratic equation, for which it easy to check that the product of roots is negative. Since we have shown there is one real positive root, the other must be real and negative, so there is only one positive solution.
Note that this solution does not depend on anything stochastic, and it is constant. So the optimal plan is to immediately set $\hat{\sigma}^{2}$ to its optimal value and keep it there forever. However, we have to check this solution for feasibility. The solution value can be arbitrarily large. From the FOC above one can show that as the information cost $\phi$ goes to infinity, the optimal ratio $\phi / \hat{\sigma}_{t}^{2} \rightarrow$ $\sigma_{\varepsilon}^{2} /(1-\beta)$, so that the optimal $\hat{\sigma}_{t}^{2}$ increases without bound as $\phi \rightarrow \infty$. If the solution value for $\hat{\sigma}_{t}^{2}$ exceeds the variance of the distribution of initial uncertainty about $z$, the optimal plan is to collect no information, simply setting $\hat{z}_{t}=\rho^{t} \hat{z}_{0}$. Since no information is being collected, uncertainty about $z_{t}$ may
increase over time. (It could also decrease, if the initial distribution for $z_{0}$ had variance greater than the steady-state unconditional variance $\sigma_{\varepsilon}^{2} /\left(1-\rho^{2}\right)$.) If $\rho^{2} \sigma_{t-1}^{2}+\sigma_{\varepsilon}^{2}$ does eventually rise above the solution value, at that point it is optimal to switch to the solution value for $\hat{\sigma}_{t}^{2}$ and keep it constant thereafter.
(b) Is it true that if $\theta$ and $\phi$ are chosen appropriately, the rational inattention and adjustment cost solutions imply the same joint stochastic behavior for $z$ and $x$ ? (You should be able to answer this even if you can't get an explicit solution for the RI model.) Explain your answer.
No. In the RI solution, the agent never learns exactly the value of $z_{t}$, so there is no exact functional relationship between $z_{t}$ and $x_{t}$ conditional on $x_{t-1}$ There are equivalence results between adjustment cost RE models and RI models, but they involve models where the agents are constrained to observe the state (here $z_{t}$ ) only with error. In the RE model above, the agents observe the state without error, and only damping of their reaction to the state, not randomness in their reaction, arises from the adjustment cost.
(2) Suppose we take the standard "NK Phillips curve" and "NK IS curve" and combine them with an active-fiscal/passive-money description of policy. That is, we consider a system like this:

$$
\begin{align*}
\pi_{t} & =\theta E_{t} \pi_{t+1}+\phi c_{t}+\varepsilon_{t}  \tag{4}\\
r_{t} & =\gamma\left(E_{t} c_{t+1}-c_{t}\right)+E_{t} \pi_{t+1}+\rho+v_{t}  \tag{5}\\
b_{t} & =\left(1+r_{t-1}-\pi_{t}\right) b_{t-1}-\left(\tau_{t}+\xi_{t}\right)  \tag{6}\\
r_{t} & =\rho+\zeta_{t} \tag{7}
\end{align*}
$$

where $\pi$ is inflation, $c$ is $\log$ consumption (and output), $r$ is the nominal interest rate, and $b$ is real one-period government debt, The first two equations are fully linearized. To proceed you will first need to get the whole system, including the third equation, linearized around a steady state, which you can assume makes steady state $\operatorname{debt} \bar{b}=1$. Assume all the shocks appearing at the ends of the equations are i.i.d., zero mean. For parameter values, use $\gamma=2, \phi=.5, \theta=.7, \rho=.05$.
(a) Using a linear rational expectations system solver (e.g. gensys), determine how this system behaves. In particular, are increased deficits (reductions in $\xi_{t}$ ) expansionary? Inflationary? Are monetary-policyinduced decreases in interest rates (decreases in $\zeta_{t}$ ) expansionary? Inflationary?
The $t$ subscript on $\tau$ was a mistake. If you treated it as an endogenous variable, there were not enough equations to determine all the variables. If you treated it as a shock, it was equivalent to $\xi_{t}$. It should have been treated, and is so treated in my answers below, as a parameter with a constant positive value. In order for the steady-state $b$ to be one as stated, $\tau$ has to be equal to $\rho$. It should have been clear from the statement of the problem, where an
active-fiscal/passive-money policy configuration was specified, that $\tau_{t}$ either was exogenous or required an additional equation (with $\tau$ responding weakly to $b$ ) to complete the model. Attached is a printout of R code showing a solution of the model. The only somewhat non-standard consideration is that there are shocks in the IS and Phillips curve equations that are dated $t$. In gensys notation, where the forward-looking notations are shifted back in time, that means that those shocks are known one period in advance. The impact matrix shows that they have no contemporaneous impact. If they are i.i.d. and mean zero (as is assumed in the solution I show), then their effect at time $t$ is the first two columns of the first coefficient matrix in the forwardlooking part of the solution, i.e. ywt $\% * \%$ fmat $\% * \%$ fwt. Combining these two columns with the last two columns of impact, we get the full matrix of contemporaneous effects of shocks, which I label AO in the code below. The impulse responses are then the sequence of matrices A0, G1 $\% * \% \mathrm{~A} 0, \mathrm{G} 1$ $\% * \%$ G1 $\% * \%$ A0, ... They show that deficits, or reduced primary surpluses, are indeed expansionary, increasing both inflation and output. Monetary policy induced interest rate declines, though, decrease inflation and, except for a small increase in the initial period, also decrease output.
(b) In simple flexible-price models, a $1 \%$ increase in all expected future primary surpluses with the current level of nominal debt unchanged must be fully offset by a $1 \%$ decrease in the current price level. That is not true in this model. Why not?
That result depends on future surpluses being discounted at a fixed rate. In this model, $\rho$ is just the steady-state real rate. The actual real rate depends on the growth rate of output. In this Keynesian model, the change in expected future surpluses affects the current level of output and its expected growth rate, and thereby the real discount rate applied to future surpluses to arrive at the real value of debt. Another way to put it is that the flex-price adjustment mechanism, in which the price level jumps while future inflation rates and real variables are unchanged, is not possible here because of the Phillips curve

```
> exmg0g1
$g0
\begin{tabular}{lrrrr} 
& pi & c & r & \(b\) \\
Phillips & -0.7 & 0 & 0 & 0 \\
IS & -1.0 & -2 & 0 & 0 \\
bc & 2.0 & 0 & 0 & 1 \\
mpol & 0.0 & 0 & 1 & 0
\end{tabular}
$g1
_rrrer
IS }\quad0\quad-2.0 -1 0.0
bc 0}00.0\quad21.0
mpol 0}00.0000.0
```

\$Psi

|  | eps | nu | xi | zeta |
| :--- | ---: | ---: | ---: | ---: |
| Phillips | 1 | 0 | 0 | 0 |
| IS | 0 | 1 | 0 | 0 |
| bc | 0 | 0 | -1 | 0 |
| mpol | 0 | 0 | 0 | 1 |


| \$Pi |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
|  | Phillips | IS |  |  |
| Phillips | -1 | 0 |  |  |
| IS | 0 | -1 |  |  |
| bc | 0 | 0 |  |  |
| mpol |  | 0 | 0 |  |
|  |  |  |  |  |
| \$param |  |  |  |  |
| theta | phi | gamma | rho | tau |
| 0.70 | 0.30 | 2.00 | 0.05 | 0.10 |

```
> args(gensys)
function (g0, g1, c0 = matrix(0, dim(g0)[1], 1), psi, pi, div = -1)
NULL
> with(exmg0g1, gensys(g0,g1, c0=matrix(c(0, .05, .1, .05), ncol=1), psi=Psi, pi=Pi))
$G1
    pi c r b
pi -3.195008e-18+0i 5.057126e-17+0i 2.782222e-01+0i 1.460667e-01+0i
c 5.021211e-18+0i 3.363398e-17+0i 4.354116e-01+0i 2.285911e-01+0i
r -3.103117e-17+0i 2.091316e-16+0i -4.841086e-18+0i -5.709845e-17+0i
b 1.970537e-17+0i 1.517205e-17+0i 1.443556e+00+0i 7.578666e-01+0i
```

\$C
[,1]
pi $0.5703556+0$ i
c $0.8925938+0 i$
r $0.0500000+0 i$
b -1.0407112+0i
\$impact

|  | $e p s$ | nu | xi | zeta |
| :--- | ---: | ---: | ---: | ---: |
| pi | $1.348800 \mathrm{e}-17+0 \mathrm{i}$ | $-2.966372 \mathrm{e}-17+0 \mathrm{i}$ | $-1.391111 \mathrm{e}-01+0 \mathrm{i}$ | $0.15670690+0 \mathrm{i}$ |
| c | $1.020662 \mathrm{e}-17+0 \mathrm{i}$ | $-2.502146 \mathrm{e}-17+0 \mathrm{i}$ | $-2.177058 \mathrm{e}-01+0 \mathrm{i}$ | $-0.02001051+0 \mathrm{i}$ |
| r | $4.109958 \mathrm{e}-17+0 \mathrm{i}$ | $-8.068367 \mathrm{e}-17+0 \mathrm{i}$ | $4.607861 \mathrm{e}-17+0 \mathrm{i}$ | $1.00000000+0 \mathrm{i}$ |
| b | $2.368431 \mathrm{e}-17+0 \mathrm{i}$ | $-6.302105 \mathrm{e}-17+0 \mathrm{i}$ | $-7.217778 \mathrm{e}-01+0 \mathrm{i}$ | $-0.31341379+0 \mathrm{i}$ |

\$fmat
[,1] [,2]
[1,] 9.523810e-01+0i $0.3413459+0 \mathrm{i}$
[2,] 1.720847e-18+0i $0.5305067+0 i$
\$fwt

|  | eps | nu | xi | zeta |
| ---: | ---: | ---: | ---: | ---: |
| $[1]$, | $-0.4803045+0 \mathrm{i}$ | $-0.4695669+0 \mathrm{i}$ | $3.165479 \mathrm{e}-01+0 \mathrm{i}$ | $-0.10747875+0 \mathrm{i}$ |
| $[2]$, | $0.8341935+0 i$ | $-0.1413902+0 i$ | $4.809532 \mathrm{e}-19+0 \mathrm{i}$ | $0.07500846+0 \mathrm{i}$ |

\$ywt

|  | $[, 1]$ | $[, 2]$ |
| :--- | ---: | ---: |
| pi | $-4.185363 \mathrm{e}-01+0 \mathrm{i}$ | $6.242591 \mathrm{e}-01+0 \mathrm{i}$ |
| c | $-6.550001 \mathrm{e}-01+0 \mathrm{i}$ | $-8.990854 \mathrm{e}-01+0 \mathrm{i}$ |
| r | $1.241683 \mathrm{e}-16+0 \mathrm{i}$ | $9.528782 \mathrm{e}-17+0 \mathrm{i}$ |
| b | $8.370726 \mathrm{e}-01+0 \mathrm{i}$ | $-1.248518 \mathrm{e}+00+0 \mathrm{i}$ |

\$gev

|  | $[, 1]$ | $[, 2]$ |
| ---: | ---: | ---: |
| $[1]$, | $2.255482+0 i$ | $1.709354+0 i$ |
| $[2]$, | $1.000000+0 i$ | $0.000000+0 i$ |
| $[3]$, | $1.000000+0 i$ | $1.050000+0 i$ |
| $[4]$, | $0.620710+0 i$ | $1.170033+0 i$ |

## \$eu

[1] 11
\$loose

```
            [,1] [,2]
pi -1.541465e-16+0i -7.167529e-17+0i
c 1.738671e-16+0i -1.767026e-16+0i
r -2.262437e-17+0i -1.812480e-17+0i
b 2.934644e-16+0i 4.916957e-17+0i
> gout <- .Last.value
> with(gout, ywt %*% fmat %*% fwt)
```

|  | eps | nu | xi | zeta |
| :--- | ---: | ---: | ---: | ---: |
| pi | $3.485375 \mathrm{e}-01+0 \mathrm{i}$ | $1.605473 \mathrm{e}-01+0 \mathrm{i}$ | $-1.261779 \mathrm{e}-01+0 \mathrm{i}$ | $5.696637 \mathrm{e}-02+0 \mathrm{i}$ |
| c | $-2.847776 \mathrm{e}-01+0 \mathrm{i}$ | $3.919716 \mathrm{e}-01+0 \mathrm{i}$ | $-1.974656 \mathrm{e}-01+0 \mathrm{i}$ | $1.449892 \mathrm{e}-02+0 \mathrm{i}$ |
| r | $2.072725 \mathrm{e}-17+0 \mathrm{i}$ | $-6.866899 \mathrm{e}-17+0 \mathrm{i}$ | $3.743352 \mathrm{e}-17+0 \mathrm{i}$ | $-5.739029 \mathrm{e}-18+0 \mathrm{i}$ |
| b | $-6.970751 \mathrm{e}-01+0 \mathrm{i}$ | $-3.210946 \mathrm{e}-01+0 \mathrm{i}$ | $2.523558 \mathrm{e}-01+0 \mathrm{i}$ | $-1.139327 \mathrm{e}-01+0 \mathrm{i}$ |

> Re(.Last.value)

|  | eps | nu | xi | zeta |
| :---: | :---: | :---: | :---: | :---: |
| pi | $3.485375 \mathrm{e}-01$ | $1.605473 \mathrm{e}-0$ | $-1.261779 \mathrm{e}-01$ | $5.696637 \mathrm{e}-02$ |
| c | -2.847776e-01 | $3.919716 \mathrm{e}-01$ | -1.974656e-01 | $1.449892 \mathrm{e}-02$ |
| r | $2.072725 \mathrm{e}-17$ | -6.866899e-17 | $3.743352 \mathrm{e}-17$ | $-5.739029 \mathrm{e}-18$ |
|  | -6.970751e-01 | -3.210946e-01 | $2.523558 \mathrm{e}-01$ | -1.139327e-01 |
| > A0 <- gout\$impact |  |  |  |  |
| > AO[ , 1:2] <- with(gout, (ywt \%*\% fmat \%*\% fwt)[ , 1:2]) |  |  |  |  |
|  |  |  |  |  |



## \$vectors

|  | [,1] | ] [,2] | [,3] | [,4] |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| [1, ] | $1.814596 \mathrm{e}-01+0 \mathrm{i}$ | i -0.2946103+0i | i 0.4596859+0i | 0.4960330+0i |  |
| [2,] | $2.839803 \mathrm{e}-01+0 \mathrm{i}$ | i 0.2345954+0i | i 0.2180159+0i | - $0.2247449+0 i$ |  |
| [3,] | -8.748735e-27+0i | i -0.4306103+0i | i -0.4001780+0i | -0.3898627+0i |  |
| [4, ] | $9.415028 \mathrm{e}-01+0 \mathrm{i}$ | i 0.8202101+0i | i 0.7622438+0i | - $0.7425956+0 i$ |  |
| > resp [ , , 1] <- A0 |  |  |  |  |  |
| > for (it in 2:10) resp[ , , it] <- gout\$G1 \%*\% resp[ , , it-1] |  |  |  |  |  |
| > resp[1, 3:4, ] |  |  |  |  |  |
|  | [,1] | [,2] | [,3] | [,4] | [,5] |
| [1, ] | -0.1391111+0i -0 | $0.1054277+0 \mathrm{i}-0$ | $0.07990012+0 i-0$ | -0.06055364+0i | -0.04589158+0i |
| [2,] | $0.1567069+0 \mathrm{i}$ | $0.2324429+0 \mathrm{i}$ | $0.17616074+0 i$ | $0.13350635+0 i$ | $0.10118001+0 i$ |
|  | [,6] | [,7] | [,8] | [,9] | [,10] |
| [1, ] | -0.03477970+0i | -0.02635837+0i | -0.01997613+0i | -0.01513924+0i | i -0.01147353+0i |
| [2,] | $0.07668096+0 i$ | $0.05811394+0 i$ | $0.04404262+0 i$ | i 0.03337843+0i | i 0.02529640+0i |
| > resp [2, 3:4, ] |  |  |  |  |  |
|  | [,1] | [,2] | [,3] | [,4] | [,5] |

[1,] $-0.21770582+0 i-0.164992+0 i-0.1250419+0 i-0.0947651+0 i-0.07181931+0 i$
[2,] -0.02001051+0i $0.363768+0 i \quad 0.2756877+0 i \quad 0.2089345+0 i \quad 0.15834447+0 i$
[,6] [,7] [,8] [,9] [,10]
[1,] -0.05442946+0i -0.04125027+0i -0.03126220+0i -0.02369258+0i -0.01795582+0i
$[2] \quad 0.12000400+,0 i \quad 0.09094703+0 i \quad 0.06892572+0 i \quad 0.05223650+0 i \quad 0.03958830+0$

