We consider a continuous time, perfect foresight, FTPL model in which the only sort of bond held by private agents or the central bank is denominated in foreign currency. The foreign bonds pay a fixed interest rate ρ. The central bank has no support from the treasury and is thus entirely reliant on seignorage and interest earnings to finance itself. Agents maximize

$$\int_0^\infty e^{-\beta t} \log(C_t) dt \quad \text{subject to}$$

$$C \cdot (1 + \gamma f(v)) + \frac{\dot{M}}{P} + \dot{F}^H = \rho F^H + Y$$

$$F^H \geq 0, \qquad M \geq 0$$

$$v = \frac{PC}{M} .$$

Defining

$$Z = \lambda / P = 1/(Mv(1 + \gamma f(v) + \gamma f'(v)v)) , \tag{$*$}$$

we can reduce the first order conditions to

$$\gamma f' v^2 = \beta + \frac{\dot{P}}{P} \tag{†}$$

$$-\frac{\dot{Z}}{Z} = \beta - \rho - \frac{\dot{P}}{P} . \tag{‡}$$

The central bank must satisfy the budget constraint

$$\dot{F}^G \leq + \frac{\dot{M}}{P} + \rho F^G , \tag{1}$$

We assume that when reserves F^G are large, the inequality above is strict and excess earnings of the central bank are turned over to the treasury. However, when the central bank reserves are low, the inequality is an equality and earnings are accumulated. Any earnings turned over to the treasury are wasted by the govenrnment, not rebated to private agents.

Suppose central bank policy is to keep M constant, that $\rho = \beta$, and that Y is constant.

(1) Show that, under mild conditions on f and γ, there is a unique equilibrium in which P and C are constant. (There may be others in which P and C are not constant.)

With M constant by policy, and P and C constant by assumption, v is constant. But then from (†) we know its unique constant value if we can solve that equation for v. A sufficient condition for this would be that $v^2 f'(v)$ is monotone. This is certainly true for f linear or for the $f = v/(1 + v)$ considered below. Then ($*$) tells us that Z is constant, since it is a function of v. Then (‡) is satisfied, because we have assumed $\rho = \beta$.

The consumer's budget constraint, with constant C, v and Y, implies constant $\dot{F}^H - \rho F^H$. This implies an unstable difference equation in F^H. Assuming a bound on consumer borrowing and invoking transversality to rule out upwardly explosive wealth at the rate β, it must be that $\dot{F}^H = 0$ and ρF^H is constant. Thus equilibrium consumption in this steady state is $(Y + \rho F^H)/(1 + \gamma f(v))$. The problem statement should probably have made explicit that the "unique" level of consumption in this steady state depends on initial F^H.

The central bank in this steady state, if it starts with positive F^G, will earn positive seignorage, which it turns over to the treasury so that F^G does not grow. The problem should have made explicit that the treasury spends this revenue in ways that neither extract resources from the economy nor provide income to private agents. This is implicit in the way the private budget constraint is written, since there are no taxes or transfers in it. We could imagine a government that uses seignorage to buy foreign-made military jet aircraft, e.g.

(2) Show that if $f(v) = v/(1+v)$, there is a range of values of γ for which there is a unique equilibrium with constant prices, but that there is also a continuum of other equilibria.

There was a serious typo in the question that creates problems here. In (‡), the $\dot{P}/P$ term on the right should enter with a plus sign, not a minus. Allowances were made for this in grading, though no one got far enough with an answer to make this a crucial point. With the sign corrected, (†) together with (‡) give us

$$-\frac{\dot{Z}}{Z} = \gamma \frac{v^2}{(1+v)^2} - \beta \,. \qquad (**)$$

since $v^2/(1+v)^2 < 1$, existence of a steady state requires $\gamma > \beta$. It is not hard to verify that Z is monotone decreasing in v. Therefore this equation states that once v goes above its steady state value, it continues increasing indefinitely, and vice versa once it goes below its steady state value. If it goes below, $\dot{Z}/Z$ eventually starts increasing at the steady exponential rate β, so it goes to infinity. This would require $v \to 0$, but as this requires accumulation of arbitrarily large non-interest-bearing real balances while transactions costs are becoming negligible, we can rule this out as non-optimal (i.e. by transversality). But steadily decreasing Z, with v steadily increasing may be possible. As $v \to \infty$, $\gamma v^2/(1+v)^2 \to \gamma$. so if $\gamma > \beta$, there are feasible paths, consistent with equilibrium in which velocity goes to infinity and transactions costs, as a proportion of consumption, approach a finite bound. These are equilibria in which steady inflation erodes the value of real balances so that they converge to zero.

(3) Show that in every equilibrium, whether prices are constant or not, if $F^G < M/P$ at the start (so the central bank has negative net worth initially), eventually $F^G \geq M/P$.

The only equilibria we have found are ones with P constant or increasing. If M is constant as assumed, therefore, M/P is constant or shrinking. But the central bank budget constraint, under the $\dot{M} = 0$ policy, implies that F^G grows exponentially at the rate ρ. Reserves must therefore eventually exceed M/P.

(4) Does this result contradict the idea that a completely independent central bank with negative net worth faces limits on its ability to commit to controlling inflation?

A completely independent central bank can always implement a fixed-M policy, regardless of its net worth. However a fixed-M policy does not necessarily guarantee a stable price level. In this model with $f(v) = v/(1+v)$ a constant-M policy, which we have studied here, does not guarantee a stable price level. Money disappears asymptotically from the economy via inflation in many of this economy's equilibria. With $f(v) = v$ equilibria with explosive prices and velocity and real balances going to zero also exist, though in this case such equilibria imply that real balances become totally worthless in finite time. One might argue that these equilibria are so pathological people would be confident they would be eliminated by a fiscal intervention. In realistic variants of this model, in which for example γ was subject to shifts or shocks, The fixed- M policy must either accept corresponding fluctuations in P, or offset the shocks with variations in M. Its ability to do this via open market operations (the only route for changing M allowed in this model) is limited if its net worth is negative.