CONTINUOUS TIME FTPL MODEL

Agents maximize

$$\int_0^\infty e^{-\beta t}\log(C_t)dt \quad \text{subject to} \tag{1}$$

$$C\cdot(1+\gamma f(v))+\frac{\dot M+\dot B}{P}+\tau=\frac{rB}{P}+Y \tag{2}$$

$$B\geq 0, \qquad M\geq 0 \tag{3}$$

$$v=\frac{PC}{M}\,. \tag{4}$$

First order conditions are

$$\partial C: \qquad \frac{1}{C}=\lambda\cdot(1+\gamma f(v)+\gamma f'(v)v) \tag{5}$$

$$\partial B: \qquad -\frac{d}{dt}\left(\frac{\lambda}{P}\right)+\beta\frac{\lambda}{P}=r\frac{\lambda}{P} \tag{6}$$

$$\partial M: \qquad -\frac{d}{dt}\left(\frac{\lambda}{P}\right)+\beta\frac{\lambda}{P}=\gamma f'\cdot v^2\frac{\lambda}{P}\,. \tag{7}$$

Defining

$$Z=\lambda/P=1/(Mv(1+\gamma f(v)+\gamma f'(v)v))\,, \tag{8}$$

we can derive

$$\gamma f'v^2=r \tag{9}$$

$$-\dot Z+\beta Z=rZ\,. \tag{10}$$

Government must satisfy the budget constraint

$$\tau+\frac{\dot B+\dot M}{P}=r\frac{B}{P}\,, \tag{11}$$

which can be combined with the representative agent's budget constraint to produce the social resource constraint

$$C(1+\gamma f(v))=Y\,. \tag{12}$$

1. Active money

One strongly "active" monetary policy rule would be $r=\theta/Z$, for some positive constant θ. Z decreases with increases in either M or v, so on inflationary paths for the economy this rule makes the nominal interest rate rise sharply. This leads to the differential equation

$$-\dot Z+\beta Z=\theta\,, \tag{13}$$

which is a standard linear differential equation, for which the solution is well known to be of the form

$$Z = \frac{\theta}{\beta} + Ae^{\beta t}\,, \tag{14}$$

for some constant A. One solution to this equation is $Z \equiv \theta/\beta$, in which case policy makes r also constant, this implies v is constant. If we take Y to be constant also, this implies C is constant. Constancy of Z and v implies M is constant, and then finally P also must be constant because $v = PC/M$ is constant.

But if initial Z is above θ/β, $A > 0$ and $Z \to \infty$, while if initial Z is below θ/β, $A < 0$ and $Z \to 0$ in finite time. Are these solutions to the differential equation equilibrium paths?

1.1. $\mathbf{f(v) = v}$. To keep things simple, we will suppose Y is constant. Consider first the possibility that $Z \to \infty$. Suppose also for example that $f(v) = v$. Then since $r = \gamma f' v^2 = \theta/Z$, $Z \to \infty$ implies $v \to 0$. Since as $v \to 0$, $C \to Y$.the only way that $v \to 0$ is possible is for M/P to go to infinity, even though C remains bounded. This can be ruled out by the private agent transversality condition. Formal invocation of the TVC would require that we bring in the private budget constraint and an explicit fiscal policy, because the TVC will apply to total wealth $(M+B)/P$. We can postpone thinking about the budget constraint, though, with a direct argument that unbounded M/P is impossible. The argument is essentially the same as that given in section 9 of the 2005 version of the FTPL notes from ECO504. Briefly, the argument is that if M/P gets large enough, the price-taking agent will see the discounted present value of the increased transactions costs due to a given percentage reduction in M/P as becoming arbitrarily small, but the utility gains of consuming that fraction of M/P in a given short span of time will grow arbitrarily large. So persisting on a path in which M/P grows without bound cannot be optimal.

If instead Z starts low and therefore $Z \to 0$ in finite time, with our same assumption on the form of f we must have $v \to \infty$ — in finite time. It is hard to see how to rule this out as an equilibrium. At no point along the time path does any agent have an opportunity to improve welfare by deviating. Once money has become totally valueless, $C = 0$ because transactions costs absorb all output. This means utility is $-\infty$, which is certainly a bad outcome, but there is no way an individual has an incentive to deviate and improve his own situation. One might say then that starting with Z anywhere below its steady-state value does lead to an equilibrium in which money loses all its value in finite time.

As we discussed in class, if the government were believed to be ready to tax to back the value of fiat money if that value became very low, then these explosive equilibria with money disappearing would not be possible. The "fiscal commitment" that guarantees the unique solution never actually is tested in equilibrium, while the nature of monetary policy determines the price level and its time path. But this raises questions about how belief in the commitment is sustained if it is never tested.

1.2. $\mathbf{f(v) = v/(1+v)}$. Now

$$r = \gamma f' v^2 = \frac{\gamma v^2}{(1+v)^2} \,. \tag{15}$$

If we consider the same active $r = \theta/Z$ monetary policy, we get the same differential equation in Z, but now

$$r = \frac{\theta}{z} = \frac{\gamma v^2}{(1+v)^2} \,. \tag{16}$$

The right hand side is now still increasing in v, but bounded above by γ. This means that Z is bounded below by θ/γ. So solutions to the differential equation in Z that start below steady state and head downward result in infinite velocity, and money disappearing from the economy, even before Z hits zero. Now, though, it is easy to interpret the post-money economy. Without money, transactions costs just shrink output by the factor $1/(1+\gamma)$ as it is converted from Y to C. There is therefore a barter equilibrium with utility bounded away from zero, and the economy simply enters that equilibrium when the value of money has reached zero. The same argument as before rules out the paths in which $Z \to \infty$. It is only the possibility of explosive inflation leading to a barter economy that creates indeterminacy here. Note also that the steady state Z is still θ/β. If the lower bound θ/γ on Z exceeds the steady state value, i.e. if $\gamma < \beta$, then there is no steady state equilibrium. Every value of initial Z corresponds to a path that converges to the barter equilibrium.

As before, a fiscal commitment to put a lower bound on the value of money will eliminate explosive paths as equilibria. What about the case of $\gamma < \beta$, though? A "backstop" fiscal commitment in this case eliminates *all* the solutions for the Z differential equation. The only reasonable interpretation then is that it cannot really be a never-invoked backstop. The fiscal commitment will have to be drawn on immediately, and in every period. Determining exactly what equilibrium emerges requires a more complete specification of fiscal policy.

1.3. $\mathbf{P_t = \bar{P}}$. It is worth noting that there is a form of active monetary policy that will guarantee a unique equilibrium, assuming there is a non-zero stock of outstanding government debt. The monetary authority can commit to $P_t \equiv \bar{P}$. It would do so by standing ready to exchange money for goods or vice versa at the rate $\bar{P}$. In this economy there is no real store of value, so the government would have to sell interest-bearing bonds to acquire the real goods that it offered in exchange for money, or, if it acquired real goods for money, it would have to use the goods it acquired immediately to sell back to the public in exchange for bonds. In other words, the net transactions would always simply be exchanges of bonds for money on the balance sheets of both the central bank and the private sector. Of course in a real economy this kind of price peg is impractical, because there are many goods, most of which are not nearly as liquid as government bonds, and there are competing price indexes that often move differently. But in our simple one-good model a price peg is feasible. It would imply endogenously (not as a monetary policy commitment) that $r = \beta$, $\dot{Z} = 0$, $\dot{v} = 0$ and $\gamma f' v^2 = \beta$. There would be constant real balances, constant price level, and constant money stock. The only interesting action would be instantaneous, in the first period. Letting $\bar{v}$ be the velocity that satisfies the $\gamma f' v^2 = \beta$ condition, we must have

$$\frac{\bar{P}C}{M} = \bar{v} = \frac{\bar{P}Y}{M(1+\gamma f(\bar{v}))}\,. \tag{17}$$

This equation uniquely determines equilibrium M. Therefore in the initial period, the price peg will require a jump in M to the equilibrium level, if it is not already there. This will occur via open market operations.

With a constant price level and $r = \beta$, the government budget constaint will imply explosive growth of real debt, violating transversality, unless the primary surplus τ moves with the level of real debt. This requires passive fiscal policy. But any passive fiscal policy will work.

Other than the practical limitations cited above for a multi-good economy, the only limitation on this policy is that $\bar{P}$ must not be chosen so high that the required equilibrium level of M cannot be achieved even if the government issues money to buy the entire stock of real debt in the hands of the public.

2. Passive money

This is the case actually studied in class. It involves a government commitment to keep r constant, increase it only slightly, or even decrease it, in response to inflationary pressure. In class we looked at a $r = \theta z$ policy, which leads to convergence of Z to steady state from any initial condition. Since this policy involves decreasing r in response to inflationary pressure, it is perhaps more realistic to consider $r = \bar{r}$ as a policy. This leads to

$$\frac{\dot{Z}}{Z} = \beta - \bar{r} : . \tag{18}$$

In other words, Z grows exponentially from any initial condition, either increasing, remaining constant, or decreasing, depending on whether $\bar{r} < \beta$, $\bar{r} = \beta$ or $\bar{r} > \beta$. This would seem to leave the equilibrium indeterminate, and in older textbooks you will see the assertion that a fixed-r monetary policy leaves the price level indeterminate.

However, we have to consider whether fiscal policy can be such as to allow real debt to be stable with these Z paths. If fiscal policy is active, for example $\tau \equiv \bar{\tau}$, There is a unique equilibrium. To see this, note that with r constant, v must be constant, because of the liquidity preference relation (9), and therefore, because of the social resource constraint (12), also C is constant. From the definition of Z (8), then, it is clear that M grows at $\bar{r} - \beta$: minus the exponential growth rate of Z. Then for v to be constant, M must also be growing at this rate $\bar{r} - \beta$. The government budget constraint then can be written as

$$(\bar{r} - \beta)\frac{M}{P} + \frac{d}{dt}\left(\frac{B}{P}\right) + (\bar{r} - \beta)\frac{B}{P} + \bar{\tau} = \bar{r}\frac{B}{P} \tag{19}$$

or, rearranging terms

$$\frac{d}{dt}\left(\frac{B}{P}\right) = -(\bar{r} - \beta)\frac{Y}{\bar{v}(1+\gamma f(\bar{v}))} + \beta\frac{B}{P} - \bar{\tau}\,. \tag{20}$$

This is an unstable differential equation in B/P. Explosion upward is ruled out by transversality conditions of the private sector. Explosion downward is ruled out by the condition that $B > 0$. Because agents can't borrow from the government, they can't sustain the consumption and tax spending implied by these paths with downward explosive B/P. The only stable solution is that with

$$\frac{B}{P} = \frac{\sigma + \tau}{\beta}\,, \tag{21}$$

where $\sigma = (\bar{r} - \beta)Y/\big(\bar{v}(1 + \gamma f(\bar{v}))\big)$ is the seignorage term in the budget constraint. This determines B/P, but that is still not quite enough to pin down the price level because initially it is $B + M$, total liabilities of the government, not B alone, that is predetermined. However, the fixed interest rate does pin down velocity and hence $m = M/P$, so if initial $B + M = A_0$, we have $B + mP = A_0$, a second equation in B, P, and constants. That equation together with (21) can be solved for a unique initial P and initial B. Of course this will imply a unique initial Z and thus via our differential equation for Z a unique time path for Z.