

EXERCISE DUE MONDAY, MARCH 10

We will make accuracy checks on solutions of the (almost) linear-quadratic permanent income model. The model assumes an agent that maximizes

$$E \left[\sum_{t=0}^{\infty} (C_t - \frac{1}{2}C_t^2)\beta^t \right]$$

subject to, for each $t = 1, \dots, \infty$,

$$W_t = R(W_{t-1} - C_{t-1}) + Y_t C_t \leq W_t. \quad (*)$$

We assume that Y_t is an exogenously given stochastic process, with Y_t i.i.d. and distributed uniformly over the interval $[0, .1]$ for every t . We also assume $R = \beta^{-1} = 1.05$.

It is conventional to study this model with (*) replaced by

$$\beta^t W_t \xrightarrow{t \rightarrow \infty} 0. \quad (\dagger)$$

This constraint (because it rules out W exploding upward) makes the transversality condition unnecessary, and it leads to a linear decision rule, with C_t a linear function of W_t . Also, it is easily shown that it leads to both C and W following random walks.

With the non-negativity constraint (*), however, we instead get a highly nonlinear optimal decision rule. Consumption that exceeds the satiation level ($C = 1$) is obviously a mistake in this case. It occurs in the conventional solution because the constraint (\dagger) requires dissaving if wealth climbs too high.

The exercise is to compute both Marcet-den Haan and “Judd” style accuracy checks for three possible decision rules for this model. The idea is to see whether they give clear indications of the inaccuracy of the solution and to extract any available insight about which direction to go in improving the solution. The three decision rules you are to check are:

$$C_t = \min \{ \beta(\bar{Y} + (1 - \beta)W_t), W_t \} \quad (1)$$

$$C_t = \min \{ \beta(\bar{Y} + (1 - \beta)W_t), W_t, 1 \} \quad (2)$$

$$C_t = \text{your guess of a function of } W_t \text{ that improves on the two above.} \quad (3)$$

Note that the first two of these are, respectively, the solution to the problem with (\dagger) replacing (*), modified to make (*) nonetheless satisfied, and a solution that further modifies the conventional solution so that it never implies borrowing. The first will generate stationary stochastic simulated paths in the limit as the length of the simulated path increases. The second will generate eventual convergence to a constant $C_t = 1$ (though, unlike the optimal solution, it will not necessarily keep C at 1 forever after the first time it reaches 1).

In generating simulated paths to apply the Marcet-den Haan approach, it is probably best to restart many times from, say, $W_{-1} = 0$ as an initial condition, rather than running a long simulation. Why?

It is part of the problem for you to decide which W_{t-1} values to look at in applying the Judd approach.

1. ANSWERS

The rules were meant to be modeled on the optimal solution to the usual LQ setup, where there are no bounds on C or W , just limits on rates of growth of the solution. But they were misstated from this point of view. The correct statement of the standard LQ solution is

$$C_t = (1 - \beta)W_t + \beta\bar{Y}.$$

The original problem statement misplaced parentheses, and the resulting rule generated explosive solutions, even in the case of rule (1), which was meant to generate random-walk-like solutions. The analysis I describe here use the corrected rules. Because the uncorrected rules generate explosive paths for C with rule rule (1), the Euler equation errors eventually blow up in simulations, so the Marcet-DenHaan simulation approach, which assumes they have well-defined stationary behavior, does not apply. Some students' answers also used a corrected form of the rules.

Another anomaly in the original problem statement gave the distribution of Y as i.i.d. $U(0, .1)$. This implies that starting from $W_0 = 0$, with high probability it will take *many* periods (thousands and thousands) before any examples of $C_t \geq 1$ are observed with either of the two rules. This answer uses the $\bar{Y} = .1$ specification as in the problem statement, though scaling up the shocks to make the simulations work quicker was OK.

The Euler equation in this problem reduces to

$$C_t = E_t C_{t+1} - \mu_t,$$

where μ_t is a Lagrange multiplier that in an optimal solution is non-negative, and non-zero only if $C_t = W_t$ so the constraint $C_t \leq W_t$ is binding. For both rules (1) and (2), $C_t = W_t$ only when $W_t < \bar{Y}$. In these cases we need to be sure that μ_t remains positive. However $C_t = W_t$ implies $W_{t+1} = Y_{t+1}$. Under our assumption of uniformly distributed i.i.d. Y_t , we can therefore derive

$$\begin{aligned} E_t[C_{t+1} | C_t = W_t] &= P[Y_{t+1} < \bar{Y}]E[Y_{t+1} | Y_{t+1} < \bar{Y}] + P[Y_{t+1} \geq \bar{Y}]E[(1 - \beta)Y_{t+1} + \beta\bar{Y} | Y_{t+1} \geq \bar{Y}] \\ &= .5 \cdot .5\bar{Y} + .5 \cdot ((1 - \beta)1.5\bar{Y} + \beta\bar{Y}) = (1 - .25\beta)\bar{Y}. \end{aligned}$$

Thus if $(1 - .25\beta)\bar{Y} < W_t < \bar{Y}$, rules (1) and (2) push consumption too high. For lower values of W_t , it is likely that W will be considerably higher next period, and thus that consumption will be higher. But when W_t is close to \bar{Y} , yet below it, then the chance of $W_{t+1} = Y_{t+1} > \bar{Y}$ is almost the same as the chance of $W_{t+1} < \bar{Y}$, and for $W > \bar{Y}$ we apply the usual linear rule, which makes C increase with W only with coefficient $1 - \beta$. This asymmetry makes the expected rate of change of consumption negative when W_t is in this range and we apply the $C_t = W_t$ component of these two proposed rules.

Note that this is in contrast to what I said on this point in class: I had the impression that there could be no conflict with the requirement that μ_t be positive, even though one student raised the possibility.

When $\bar{Y} < W_t < 2\bar{Y}$, we will have $\mu_t = 0$ but an Euler equation residual with non-zero expectation, because $E_t C_{t+1} < C_t$. This is because the rules we are considering ignore the

possibility that the constraint will bind and pull C_{t+1} below the value suggested by the standard linear rule.

When $W_t > 2\bar{Y}$, there is no chance that $W_{t+1} = W_t + Y_{t+1} - \bar{Y}$ is below \bar{Y} . Therefore for the first rule

$$\begin{aligned} C_{t+1} &= (1 - \beta) \left(R(W_t - ((1 - \beta)W_t + \beta\bar{Y})) + Y_{t+1} \right) + \beta\bar{Y} \\ &= (1 - \beta)W_t + (1 - \beta)(Y_{t+1} - \bar{Y}) + \beta\bar{Y}. \end{aligned}$$

It is easy to see that this implies $E_t C_{t+1} = C_t$.

For the second rule, the same condition holds, so long as there is no chance the linear rule component would imply $C_{t+1} > 1$. This in turn is guaranteed if $W_t < (1 - \beta\bar{Y})/(1 - \beta)$. For the original $U(0, .1)$ specification, this condition is met for $W_t < 19.95$. With the $U(0, 1)$ specification it is met for $W_t < 10.5$. Note that for W_t large enough, not only will we have $C_t = 1$ with rule (2), but we will also have $C_{t+1} = 1$ with probability 1. This happens for $W_t > 20.0475$ for $\bar{Y} = .05$ and for $W_t > 11.475$ with $\bar{Y} = .5$.

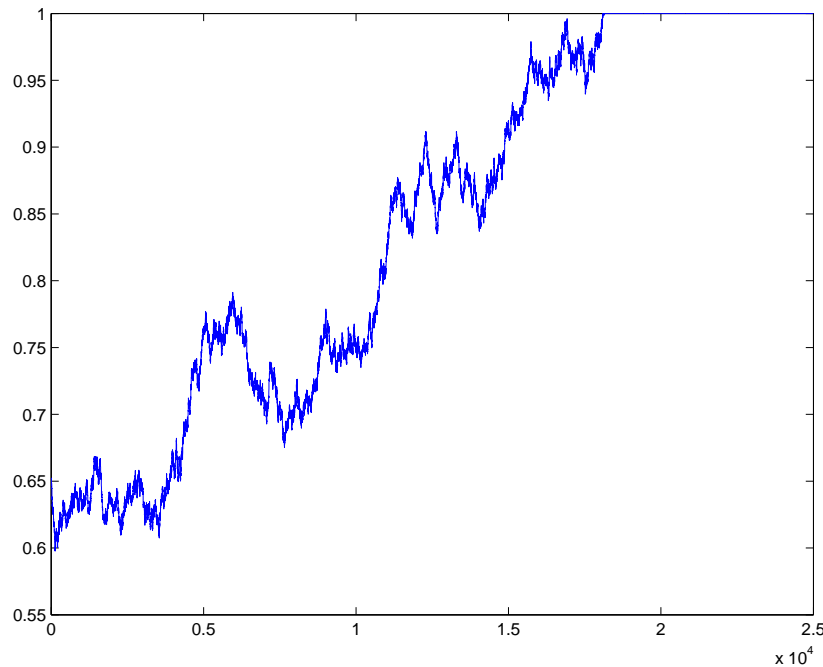
Note that the first rule, by pushing C_t above one eventually, will generate negative values for λ_t .

Marcet and Den Haan don't give separate consideration to the possibility of violations of sign constraints on Lagrange multipliers. If we check sign constraints on the Lagrange multipliers, there will be violations for both rules also in the narrow interval $(.038, .05)$. (where μ will be negative). The first rule, which does not impose $C_t \leq 1$, will generate $\lambda_t < 0$ for all $W_t > 20$. (for $\bar{Y} = .05$) or $W_t > 11$ (for $\bar{Y} = .5$).

The ranges of W values for which there are errors in the Euler equations in this setup are narrow. The lower range occurs rarely in simulations, though early if we start with $W_0 = 0$. The upper range will not occur for tens of thousands of periods in a simulation started from $W_0 = 0$. So this answer sheet will not try to calculate error measures based on simulations for these two rules. Instead we show a plot of the time paths of a typical simulations. Figure 1 shows the final 25,000 values of C_t in a simulation run of 825000 periods for rule (2), which rules out $C_t > 1$. At no time before the period shown on this graph did C hit 1. Clearly C stuck at 1 permanently within a relatively few periods after it first hit 1. The reason can be seen in Figure 2 which shows W_t over the period when C was approaching, then sticking at, 1. W fluctuated in the critical interval $(19.95, 20.0475)$ for only about 50 periods, then began shooting off exponentially to infinity, never to return.

If we are checking Lagrange multiplier signs, the first rule will start to generate sign violations for λ at around observation 818,080, when the second rule starts generating $C = 1$ occasionally and the first rule will therefore differ, by making $C > 1$. Because with rule 1 C behaves like a random walk for large W , there are likely to be many large negative values for λ_t after this point. But for the first 818,000 observations, the two rules generate the same time paths, and thus no violations of $\lambda_t > 0$.

Violations of $\mu_t > 0$ are likely to be even rarer. For a typical run of 50,000 periods, for example, 19 observations on W were in the critical $(.038, .05)$ interval. The number of observations in the interval $(\bar{Y}, 2\bar{Y})$, where the Euler equation residual also has non-zero expectation, was in this run 122, again out of 50,000 periods.

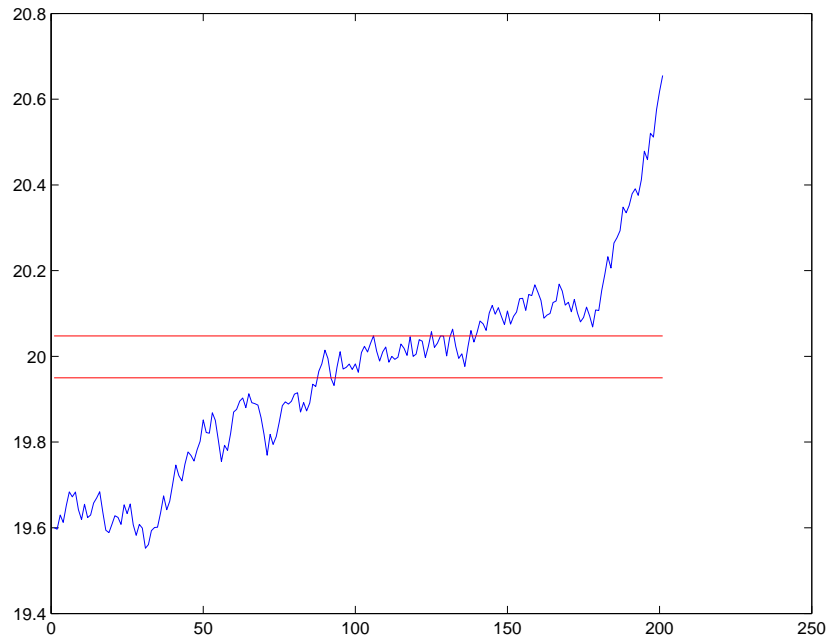
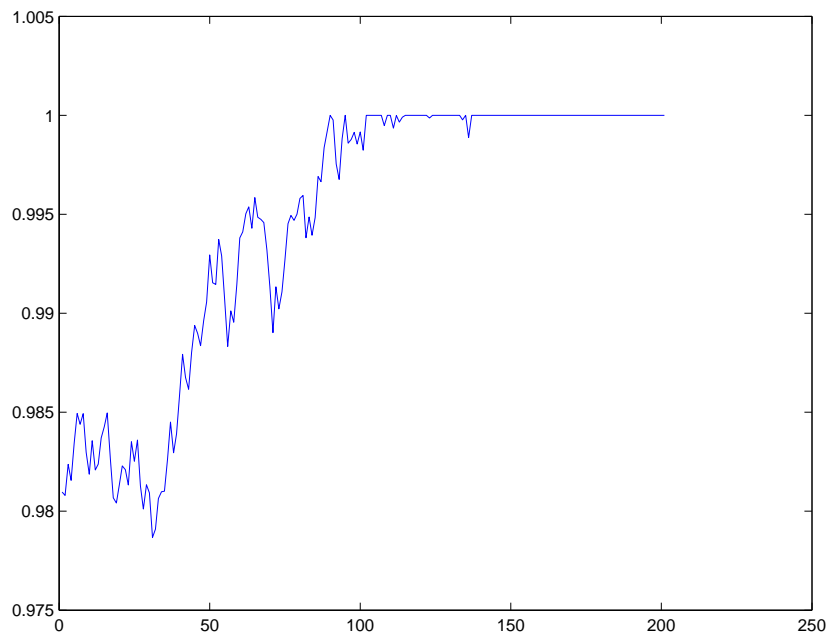
FIGURE 1. C_t for periods 800,001 through 825,000

A naive application of the Marcet Den Haan method to a long simulation would therefore have only 50 periods out of 825,000 in which the Euler equations were not perfectly accurate at the upper level of W and around 122 for which they are not perfectly accurate at the lower levels — and inside those short spans periods the errors are not very large. To get reasonable discriminatory power from this method, it would be necessary to run repeated very short simulations starting from $W_0 \doteq 19$ and/or starting from $W_0 = 0$. Of course once we start picking critical W_0 's to start from and using many short simulations, the method starts being very close to Judd's.

The code for the simulations is below

```
w0=0;
y=rand(N,1)*.1;
for it=1:N
w(it)=w0;
c(it)=min([w0,(1-1/1.05)*w0+ybar/1.05,1]);
w0=1.05*(w0-c(it))+y(it);
end
```

Plots of expected Euler equation errors in the narrow ranges where they are known to be non-zero are shown in Figures 4 and 5. Both rules give the same Euler equation errors in the low range of W 's, and for the upper range only rule (2) generates any error. So there is only a single error level to plot in each graph. In each case two lines are plotted, though. Each uses an independently generated sequence of 1000 draws for Y_t . Each line is smooth, because the same Y sequence is used to evaluate expected error for each W value along a given line. The

FIGURE 2. W_t for periods 818001 through 818200FIGURE 3. C_t for periods 818001 through 818200

differences between the two lines give an idea of how much random simulation error there is in the size of the errors.

As a third decision rule, I tried using a cubic spline over the interval $(19.5, 21)$, constrained to have slope zero at 21 and slope $1 - \beta$ at 19.5, and to have $C = 1$ at $W = 21$ and C equal

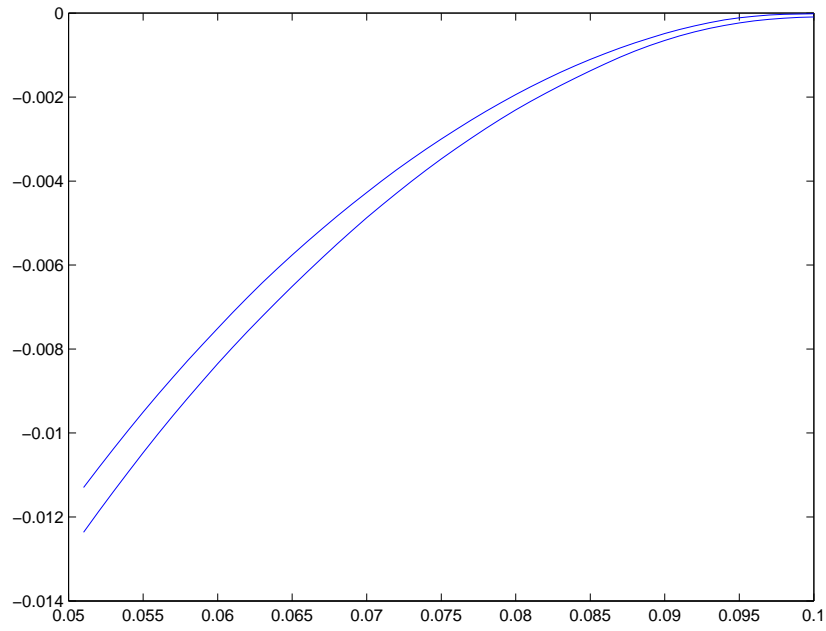


FIGURE 4. Expected Euler equation error, low W

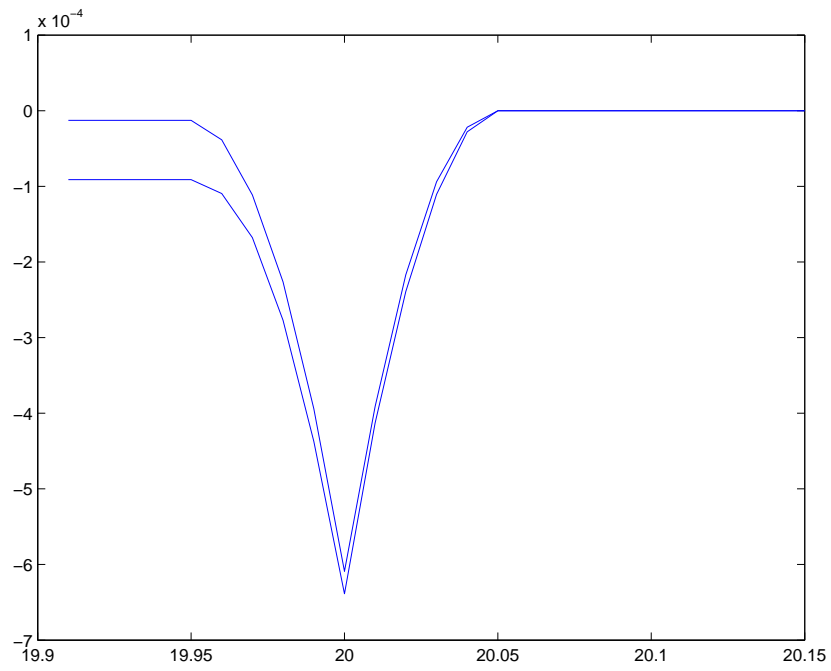


FIGURE 5. Expected Euler equation error, high W

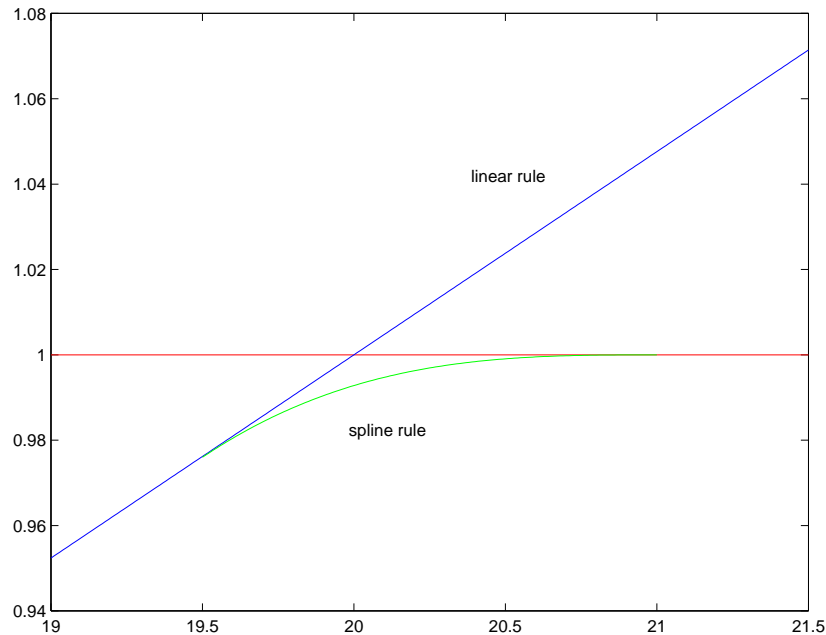


FIGURE 6. Consumption rules

to the value from the standard linear rule at $W = 19.5$. This rule gave errors over the (19,21) interval that were inside the $\pm .0001$ range that reflects the apparent Monte Carlo sampling error with 1000 draws and the rounding error in the polynomial coefficients. These errors are therefore not plotted. The spline rule itself is plotted with the other two in Figure 6.

Code that computes the Euler equation errors on a grid of W 's for the spline rule is below.

```

wa=19.5;
wb=21;
M=[wa^3 wa^2 wa 1;3*wa^2 2*wa 1 0;
wb^3 wb^2 wb 1;3*wb^2 2*wb 1 0];
r=[.976;.047619;1;0];
b=M\r;
eqe=zeros(1,151);N=1000;
for ie=1:151,c0=b(1)*w0(ie).^3+b(2)*w0(ie).^2+b(3)*w0(ie)+b(4);
for id=1:N
w1(id)=w0(ie)+y(id)-.05;
c1(id)=b(1)*w1(id).^3+b(2)*w1(id).^2+b(3)*w1(id)+b(4);
end
eqe(ie)=mean(c1-c0);
end
plot(w0,eqe)

```

For rule (2) the code is similar, but the lines calculating $c1(id)$ and $c0$ have on the right $\min((1-1/1.05)*w0(ie)+.05/1.05)$ in the case of $c0$, and the corresponding formula using $w1(id)$ in the case of $c1(id)$.